PROCEEDINGS

OF THE

46th CONFERENCE

of the International Group for the
Psychology of Mathematics Education

Haifa, Israel | July 16 – 21, 2023

PME-46
MATHEMATICS EDUCATION
FOR GLOBAL SUSTAINABILITY

Volume 1

Editors: Michal Ayalon, Boris Koichu, Roza Leikin, Laurie Rubel and Michal Tabach

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Volume 1

Plenary Lectures
Working Groups
Seminar
Colloquium
National Presentation
Oral Communications
Posters

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Cite as:

Michal Ayalon, Boris Koichu, Roza Leikin, Laurie Rubel & Michal Tabach (Eds.). (2023) Proceeding of the 46th Conference of the International Group for the Psychology of Mathematics Education (Vol. 1). University of Haifa, Israel: PME

Website: https://pme46.edu.haifa.ac.il/
The proceedings are also available via
https://www.igpme.org/publications/current-proceedings/

ISSN: 0771-100X
ISBN: 978-965-93112-1-7

Printed by the University of Haifa

Logo Design: Sharon Spitz (https://www.sharonspitz.com/)
Cover design: Sharona Gil (Facuty of Education, University of Haifa)

PREFACE

We welcome you to PME-46, the 46th Annual Conference of the International Group for the Psychology of Mathematics Education. The PME conference is one of the most significant international conferences in the field of mathematics education. PME is devoted to rigorous, systematic research that seeks to understand mathematical thinking, learning, and teaching. The PME community includes researchers with diverse fields of interest and expertise, including cognitive, social, and emotional components of mathematics education. This year's conference features an impressive lineup of 365 presentations, from 39 countries, from all of the continents, with an estimated attendance of approximately 430 participants.

The conference theme is "Mathematics education for global sustainability," to address various challenges facing humanity and the planet today, such as global diseases, economic disparities, social inequalities, and climate emergencies. These crises are interconnected and cannot be separated from one another. Mathematics education offers significant potential in finding solutions to these crises and working towards sustainability. As an international conference, PME is a place for presenting, discussing, formulating, and collaborating on research, especially with potential to increase opportunities for global sustainability. The conference theme aims to raise awareness about the significance of the United Nations' global sustainable goals (https://sdgs.un.org/goals).

The choice of the theme of global sustainability is particularly relevant in the context of the conference's host country, Israel, known as the birthplace of three major religions: Judaism, Islam, and Christianity. Situated atop a mountain in the multicultural city of Haifa, overlooking the vibrant Mediterranean Sea, the University of Haifa provides an ideal setting for research on mathematics education, society, and the environment. The university is committed to promoting social and environmental sustainability through research, teaching, and community partnerships. The university community is truly exceptional, including a diverse range of faculty, staff, and students that represent all segments of Israeli society. This creates a unique space where individuals study, teach, and learn together. Driven by the fusion of research and social responsibility, the university is dedicated to creating innovative learning environments, fostering a tight-knit community, and making valuable contributions to the betterment of Israeli society and the world.

Haifa is Israel's third largest city, a liberal city with an ethnically diverse population. Haifa is considered the capital of Israel's northern region. On the slopes of Mount Carmel with dense forests and overlooks wide views of the Mediterranean Sea, Haifa is a living example of a city that coexists with nature. The mountain range extends towards the sea and creates a spectacular mosaic of shades of green and blue, connected by the streams that pass from the Carmel forests, through the city's neighborhoods, to the surrounding beaches. Haifa offers a multitude of gardens, parks, public beaches, cafes, and restaurants. The city is known for its lively Middle Eastern markets and a bustling street food culture, providing participants with an opportunity to immerse themselves in diverse landscapes, beaches, and culinary delights.

Multiple papers submitted to PME-46 indicate that the theme of "Mathematics education for global sustainability" continues to hold relevance in various regions of the world. We eagerly anticipate engaging in cross-country discussions that will enable us to highlight our shared objectives. It is our great pleasure to welcome distinguished plenary speakers and panelists hailing from diverse countries, representing different educational contexts and theoretical perspectives. This wide-ranging diversity is expected to enrich our conference discussions on the theme. The PME-46 program encompasses a variety of session types, similar to previous PME conferences, including Research Reports, a Colloquium, Oral Communications, and Poster Presentations at the individual level. Additionally, there will be Research Forums, Working Groups, and a Seminar at the group level, fostering collaborative dialogue and exploration. Furthermore, a National Presentation will provide valuable insights into significant projects that integrate theory and practice in mathematics education within Israel.

The four volumes of the proceedings are organized according to types of presentations. Volume 1 contains the Plenary Lectures, Plenary Panel, Research Forums, Working Groups, Seminar, National Presentation. Oral communications and Poster Presentations abstracts. Volumes 2, 3 and 4 contain Research Reports.

The organization of PME-46 was generously supported by the administration of the University of Haifa. the Faculty of Education and the Department of Mathematics Education in the University of Haifa. The conference is leaded by three committees: The International Program Committee for PME-46, the International Committee of PME together with the PME Administrative Manager, and the Local Organizing Committee. We extend our gratitude and appreciation to all those involved for their support and dedication in making this conference a reality and thank all the people who generously contributed their time and expertise. Furthermore, we would like to extend our heartfelt appreciation to each participant of PME-46 for making the journey to Haifa and for their valuable contributions to this conference.

Our goal is to ensure that the PME-46 meeting is both scientifically and socially successful. Scientifically, we hope that you will leave with fresh ideas and new research directions. Socially, we aim to provide you with opportunities to forge new friendships and establish potential collaborations for the future. We wish you an engaging, informative, and inspiring experience throughout your participation in this conference.

Sincerely yours,

Michal Ayalon and Roza Leikin,

PME-46 Conference Chairs

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PLENARY LECTURES

PME-46

MATHEMATICS EDUCATION

FOR GLOBAL SUSTAINABILITY

LET'S TALK HISTORY…

Abraham Arcavi

Weizmann Institute of Science, Rehovot, Israel

History of mathematics can play important roles in enhancing the Mathematical Knowledge for Teaching (MKT). Several of these roles are described and exemplified in connection with the components of MKT.

INTRODUCTION

I am very grateful to Michal and Roza for inviting me to deliver the opening plenary. I am very honored by the responsibility, reminding myself that, according to what is attributed to Oscar Wilde, “You never get a second chance to make a first impression.”

We live in an era in which bleak horizons lie in wait regarding many aspects of life and especially in education. The theme of this conference - “Mathematics Education for Global Sustainability” - is an immense concern in mathematics and science education. Two other crucial themes nowadays are the enticing promises (and the subsequent perils) of artificial intelligence and the pernicious effects of inequity and social injustice. All three themes pose huge challenges to education and, through conferences like this, we aspire to harness ingenuity and creativity in research and development in order to influence and shape a viable future.

As a mathematics educator who has much more past behind than future ahead, it may not be a surprise that for today I choose to look back, and to remind us that engaging with history is not only a show of respect to what preceded us, but also a bountiful source of inspiration and ideas for looking ahead. So, let's talk history.

Allow me to begin with a piece of personal history. Exactly forty years ago, PME came to Israel for the first time. It was the 7th international congress organized by our dear colleague and friend Rina Hershkowitz, who years later was elected as PME President. The conference took place in the pastoral surroundings of Shoresh, in the hills near Jerusalem. At the time, I was in the middle of my PhD in the department in which Rina worked. I did not submit anything to PME, but I was very involved in the organization, assisting Rina wherever needed. Although there were only about 100-120 participants, the organizational chores were no less demanding than they are today. Back in 1983 there was no internet and we could only have dreamed of Conftool, Zoom or smartphones. I enjoyed my tasks as a technical assistant because they allowed me to get to know the lecturers in person and the topics of their presentations, the names of the participants and where they came from, and above all, it provided me with a unique opportunity to meet in person the founding fathers of our field. Among them were Gerard Vergnaud, the PME President at the time, and the four plenary speakers, David Wheeler, Hans

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 3-16). PME 46.

Freudenthal, Efraim Fischbein and Kathleen Hart. This was an unforgettable experience for me. Unfortunately, all of them have left us.

PME has come a long way with a very rich history worth perusing (Hershkowitz & Ufer, 2018). I myself came a long professional way since then, and exactly forty years later, after many PME experiences, here I am, humbly delivering a plenary. I am as thrilled now as I was then.

Now to another professional aspect of my personal history: the topic of my dissertation was “History of Mathematics as Component of Mathematics Teacher Background” (Arcavi, 1985). How did I end up working on this topic? My master’s degree thesis was about collecting all the didactical approaches I could find - after a comprehensive search worldwide in teacher journals and textbooks - to teach the multiplication of negative numbers. I classified these approaches into four categories, I developed learning units for each and then tested them in a large number of classrooms taught by their teachers. The research included administering achievement tests and attitude questionnaires. When writing the introduction to the MSc thesis (Arcavi, 1980), I came across a lovely paper by Martin Gardner (1977) describing fascinating episodes from the history of negative numbers. At the time, I also worked as a mathematics teacher, and, as such, I felt profoundly enlightened and excited by this article. I thought to myself that teachers should know and be inspired by the history of the mathematical topics they teach. My Master’s degree advisor, Professor Maxim Bruckheimer, who loved history as well, was excited. Thus, we embarked in the PhD adventure around "why history?", "what history?", and how to do it? for the benefit teachers. We developed mini courses based on what we called source-work collections on three topics: the history of negative numbers, the history of irrational numbers and the history of linear and quadratic equations. The collections consisted of worksheets, each of which was on a primary historical source followed by a set of guiding questions to support the reading and to call attention to particular aspects of the text. The collection included answer sheets with additional historical information (Arcavi et al., 1982, 1987, Arcavi & Bruckheimer, 2000).

The tailwind for this project was provided by the then recently established International Study Group on the Relations between the History and Pedagogy of Mathematics – HPM (Fasanelli & Fauvel, 2006). On the other hand, although Shulman’s (1986) seminal paper on pedagogical content knowledge had not been published yet, possibly some of his ideas were already “in the air”. In our academic work, we experience time and again that what we do “… is in large part the product of the environment… ideas seem to emerge *de novo* in new contexts, but can also be seen to be in the atmosphere if one looks at a broader, more enveloping context” (Schoenfeld, 1989, p. 77).

This “historical” work remained close to my mind and to my heart even when I moved to research and development in other areas of mathematics education. More than a decade and a half ago, I started to work on several projects focusing on teachers and teaching (e.g. Arcavi & Schoenfeld, 2008). Then Ronnie Karsenty joined me and we designed the VIDEO-LM Project (ADASHA is the Hebrew acronym) to promote teacher reflection while watching and discussing video recordings of authentic mathematics lessons using the six-lens framework developed ad-hoc (Karsenty & Arcavi, 2017). Encouraged by the resonance of VIDEO-LM among teachers and colleagues, we embarked on what we thought was a natural sequel, namely to launch teacher communities engaged in Lesson Study using the six-lens framework. This new project is called MATH-Value (Video Analysis and Lesson Study to Upgrade Expertise) and its Hebrew acronym is TAFNIT. This project is ongoing and a small piece of it will be presented in this conference.

The focus and the worldwide extensive theoretical and empirical work on mathematics teachers and mathematics teaching in the last decades, as well as my own experience with the projects described above, led me to recast my work on history in new terms (Arcavi, 2022). Thus, I am now addressing the possible roles history of mathematics may have on the knowledge required for teaching mathematics. For that purpose, I rely on the construct "Mathematical Knowledge for Teaching" (MKT) which has received much attention in the last two decades and which provided a framework for both the research and the practice of mathematics teaching.

MATHEMATICAL KNOWLEDGE FOR TEACHING (MKT)

Mathematical Knowledge for Teaching (MKT) is defined as the *specialized* mathematical knowledge required to practice and accomplish the work of teaching mathematics (Ball et al, 2008). Its two main components are Subject Matter Knowledge and Pedagogical Content Knowledge.

Subject Matter Knowledge consists of

- *Common Content Knowledge*: Competence with the mathematical topics (concepts, procedures and their underlying ideas); meta-mathematical ideas; the nature of mathematical activity and problem solving.
- *Specialized Content Knowledge*: Ways of presenting mathematical ideas; answers to "why" questions; resourcefulness to find appropriate examples and counter-examples; acquaintance with the nature and limitations of different representations; knowing how to link among them; knowledge of applications.
- *Horizon Content Knowledge*: "Horizontal depth" regarding the curriculum to be taught as "predecessor" knowledge of the content students may encounter in their future academic studies.

Pedagogical Content Knowledge consists of

- *Knowledge of Content and Students*: Anticipation of and sensitivity towards students' idiosyncratic ways of knowing, thinking and doing; competence with attentively listening to and interpreting student questions and their often unpredictable answers; discernment of difficulty levels; a repertoire of pedagogical resources to deal with them.

- *Knowledge of Content and Teaching*: Designing and sequencing instruction; judicious choice of appropriate tasks and problems; opportune assignment of different modes of student activity (group-worthy activities, digital labs, individual enquiry) according to the nature of the contents and their affordability.
- *Knowledge of Content and Curriculum*: Familiarity with diverse curricular approaches and proficiency in comparing and contrasting them in order to make thoughtful selections of materials for their classes; having the versatility for implementing them flexibly.

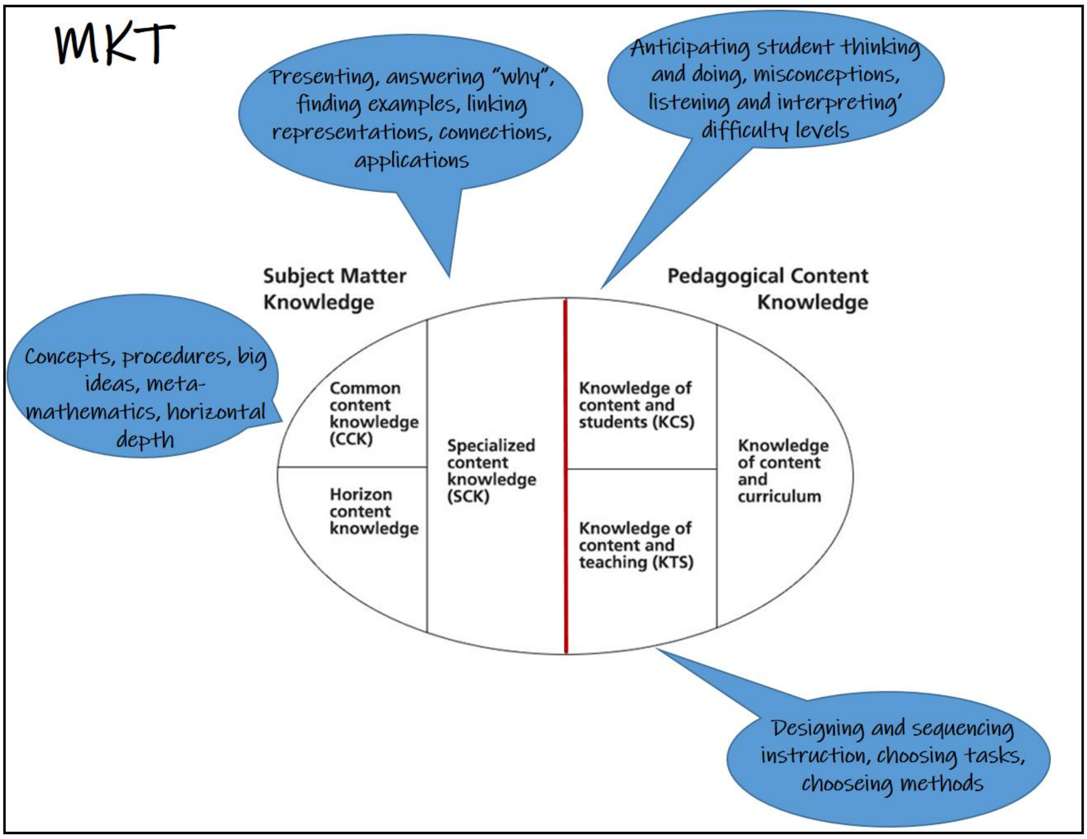

Figure 1: Graphical representation of the components of Mathematical Knowledge for Teaching

ROLES OF HISTORY OF MATHEMATICS

The history of mathematics can play several roles in supporting and enhancing mathematical knowledge for teaching (Arcavi, 2022). In the following, I focus on some of these roles:

- Ideas for tasks and problems
- Learning to listen
- Revisiting what is taken for granted
- Original texts as interlocutors

IDEAS FOR TASKS AND PROBLEMS

> "Where can I find some good problems to use in my classroom?" is a question I am often asked by mathematics teachers. My answer is simple: "the history of mathematics." (Swetz, 2000, p. 59)

> "The history of mathematics contains a wealth of material that can be used to inform and instruct in today's classroom. Among these materials are historical problems and problem solving situations" (Swetz, 2000, p. 65)

The following are just a few examples.

The Rhind Papyrus dated around 1500-1600 B.C. is one of the oldest extant mathematical documents and it became widely known less than 200 years ago. It contains arithmetic and geometric problems, some of which are intriguing even today, for example the arithmetic of unit fractions (fractions whose numerator is 1) which played an important role in ancient Egyptian arithmetic. Since doubling a number was a basic arithmetic operation, the decomposition of a fraction of the form $\frac{2}{n}$ into different unit fractions was an issue reflected in the Papyrus (e.g. Abdulaziz, 2007).

Let us explore, for example, the decomposition of $\frac{2}{9}$ into the sum of two different unit fractions. We may start with trial and error and then search for more systematic methods. For example, we can start from the obvious solution $\frac{1}{9}+\frac{1}{9}$ and proceed to replace one of the addends by the sum of different unit fractions.

We may remember and take advantage of the following general property:

$$\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)}\rightarrow\frac{1}{9}=\frac{1}{9}+\frac{1}{90}$$

And thus

$$\frac{2}{9}=\frac{1}{9}+\frac{1}{10}+\frac{1}{90}$$

We can also take advantage of this arithmetical result

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1$$

As follows:

$$\frac{1}{n}=\frac{1}{2n}+\frac{1}{3n}+\frac{1}{6n}\quad\rightarrow\quad\frac{1}{9}=\frac{1}{18}+\frac{1}{27}+\frac{1}{54}$$

Thus

$$\frac{2}{9}=\frac{1}{9}+\frac{1}{18}+\frac{1}{27}+\frac{1}{54}$$

History provides us with yet another way called the Fibonacci-Sylvester method after the famous mathematicians Fibonacci (Leonardo of Pisa) (c.1170 – c.1250) and James Joseph Sylvester (1814 – 1897). According to this method, in each step we add the largest unit fraction, which is smaller than what remains. In our case, the largest unit fraction smaller than $\frac{2}{9}$ is $\frac{1}{5}$. In this case what remains is, fortunately, a unit fraction yielding a third alternative solution: $\frac{2}{9} = \frac{1}{5} + \frac{1}{45}$.

The Egyptian table for the decomposition of $\frac{2}{n}$ (n ≤101), shows yet another possible decomposition as shown in its modern notation (Figure 2).

n	a	b	c	d
5	3	15		
7	4	28		
9	6	18		
11	6	66		
13	8	52	104	
15	10	30		
17	12	51	68	
19	12	76	114	
21	14	42		
23	12	276		
25	15	75		
27	18	54		
29	24	58	174	232
31	20	124	155	
33	22	66		
35	30	42		
37	24	111	296	
39	26	78		
41	24	246	328	
43	42	86	129	301
45	30	90		
47	30	141	470	
49	28	196		
51	34	102		
53	30	318	795	
55	30	330		
57	38	114		
59	36	236	531	
61	40	244	488	610
63	42	126		
65	39	195		
67	40	335	536	
69	46	138		
71	40	568	710	
73	60	219	292	365
75	50	150		
77	44	308		
79	60	237	316	790
81	54	162		
83	60	332	415	498
85	51	255		
87	58	174		
89	60	356	534	890
91	70	130		
93	62	186		
95	60	380	570	
97	56	679	776	
99	66	198		
101	101	202	303	606

Figure 2: The modern notation of the Egyptian table for decomposition of $\frac{2}{n}$ into $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$

We note that in order to illustrate how history can be a source of engaging problems we rely on present knowledge of algebraic symbolism. This opens up questions of historic-mathematical interest: how did the Egyptians arrive at their result? We can only speculate, since no method is provided.

The case of unit fractions is also an example of how elementary arithmetic can pose interesting mathematical issues, which may engage students and teachers in genuine mathematical activity. Thus, examples of this kind may support and enhance both Common Content Knowledge and Specialized Content Knowledge.

History also shows how ancient this mathematical topic is and yet in spite of being elementary, it includes unsolved problems as of today, such as the Erdös-Strauss conjecture proposed in 1948 and still unproven: for every $n \in N, n > 1, \frac{4}{n}$ can be expressed as the sum of three unit fractions (e.g. Graham, 2013).

Another very interesting problem from historical sources that was used in a classroom is:

> "In 1355 the Italian professor of law Bartolus of Saxoferrato (1313-1357) wrote a treatise on the division of alluvial deposits. The problem he discussed is the following… Some landowners … Gaius, Lucius and Ticius, have neighbouring properties besides the bank of a river. The river deposits silt so that the new land is formed at the riverside. How is the new fertile soil to be divided up?" (Van Maanen, 1992, p.37).

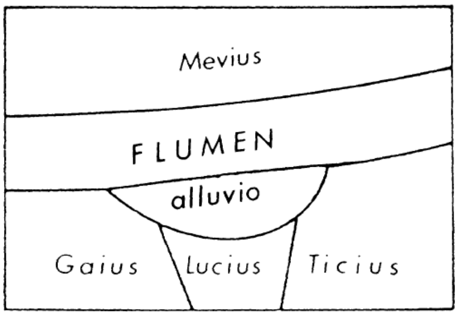

Figure 3: Map of the division of the alluvial deposit problem

The goals the teacher posed to himself for bringing this problem into the classroom were: "to demonstrate the importance of mathematics in society; …to integrate disciplines; to let pupils to discover a number of constructions with ruler and compasses… to let pupils solve some juridical problems using the constructions that they had discovered earlier" (van Maanen, 1992, p. 42).

Laurence Sherzer, a mathematics teacher in Florida, reported on an eight-grade mathematics lesson he had taught about the betweenness property of rational numbers. The class worked on methods of finding a rational number between two given rational numbers. They focused on the average and on how to calculate it (adding the two given numbers and dividing the sum by 2). The students not only practiced the procedure (adding fractions and dividing by 2), but they also noticed that by applying it repeatedly they convinced themselves that, since it is always possible to find an average between two given rational numbers, there is always one rational number between any two given ones. Then, "a student who had not been paying much attention but had been scribbling furiously suddenly interrupted. 'Sir, you don't have to go to all that trouble to find a fraction between two fractions, all you have to do is add the tops and the bottoms.'" (Sherzer, 1973, p. 229). Sherzer admitted that he was going to reject the idea outright, possibly having in mind the typical erroneous procedure many students have for adding two fractions (i.e. numerator plus numerator over denominator plus denominator). On second thoughts, he decided to go along with the student suggestion. He suggested that they go back and apply the proposed procedure to the examples they had already worked out (by finding averages), and to see whether the new method indeed yielded a number in between (not necessarily the same number obtained by the average). The class (and the teacher himself) became excited and they tried many examples, until the teacher suggested to try to find a general proof to show that the proposed procedure indeed always yields a number in between two givens. The algebraic proof is rather simple (for a visual proof, see Arcavi, 2003). The teacher acknowledged that he did not know such property and when the class was over, he "thought of that one moment when I was about to tell McKay [the student's name], 'No, that's not the way it's done'" (Sherzer, 1973, p. 230).

This property, which the teacher named as "MacKay's Theorem", was firstly documented in the book *Le Triparty en la Science de Nombres* by the French mathematician Nicolas Chuquet (1445?-1488?). The manuscript of this book remained in private hands for about four centuries and was published in 1880 in Italy. "La rigle des nombres moyens" in its English version reads as follows:

> "… whoever would want to find an intermediate between $3\frac{1}{2}$ and $3\frac{1}{3}$ should add 1 to 1, which are the numerators, and they come to 2 for numerator, and then 2 to 3 which are the two denominators, making 5 for denominator. Thus I have $3\frac{2}{5}$ as an intermediate between $3\frac{1}{2}$ and $3\frac{1}{3}$. For $\frac{2}{5}$ is more than $\frac{1}{3}$ and less than $\frac{1}{2}$." (Flegg et al., 1985, p. 91)

Had the teacher known about episodes from the history of rational numbers, and in particular Chuquet's rule (*Common Content Knowledge*), he could have saved to himself the indecision about pursuing the student suggestion, and the risk of rejecting it, that he was on the verge of doing. Fortunately, the teacher's opportune decision to follow McKay's proposal resulted in uncovering a piece of mathematics new to him and to his students (except for McKay).

LEARNING TO LISTEN

The McKay story illustrates how, in many occasions, students genuinely engage in learning and doing mathematics by relying on idiosyncratic yet productive reasoning, which is not always aligned with what teachers expect. Thus, *Knowledge of Content and Students* includes: (a) the anticipation of and the sensitivity towards student distinctive ways of knowing, thinking and doing, and (b) the competence of listening attentively and interpreting student questions, their often unpredictable ways of reasoning, their answers and their sometimes unexpected suggestions and conjectures.

By "listening to students" we mean paying careful attention to students, trying to understand what they say and do, and the possible sources and entailments thereof. Such listening should include:

- Detecting, taking up and creating opportunities for students to engage in freely expressing their mathematical ideas;
- Questioning students in order to uncover the essence and the sources of their ideas;
- Analyzing what one hears (not necessarily on the spot, sometimes in consultation with peers), making the intellectual effort to adopt the 'other's perspective' in order to understand it on its own merits;
- Deciding in which ways to integrate productively students' ideas into the development of the lesson.

The importance of listening as a teaching skill cannot be overstated. It may be a strong manifestation of "a caring, receptive and empathic form" (Smith, 2003, p. 498) of teacher-student interactions. If often modeled by teachers, students as well as their mathematical productions would feel respected and valued. Moreover, the habit of listening as modeled by the teacher may be internalized by the students and may become a habit in their own repertoire of learning techniques and interpersonal skills. Above all, listening enables teachers to better understand student thinking and incorporate it in their teaching. Moreover, listening may benefit the teachers themselves. "Thinking ourselves into other persons leads us to reflect about our own relationship to mathematics" (Jahnke, 1994, p. 155). In other words, effective listening may influence "listeners", by making them re-inspect their own knowledge against the background of what was heard from others. Such re-inspection of the listener's own understandings may promote the re-learning of some mathematics or meta-mathematics. There are several candid self-reports of this phenomenon even by mathematicians (e.g. Aharoni, 2005; Henderson, 1996).

Listening to students poses several challenges (Wallach & Even, 2005). For example, once we understand a complex idea, we may tend to forget (or even dismiss) the process we underwent while learning that idea. Listening requires unpacking that process. Listening also requires "decentering", namely the capability to adopt another person's perspective while leaving aside as much as possible our own.

In spite of the challenges it poses and given its importance, listening in order to understand the students' points of view is a learnable ability. The history of mathematics can provide rich scenarios for such learning, for example by approaching primary sources. Primary sources often offer very different ways of doing mathematics from what is common nowadays, and usually conceal the thinking behind them. When facing a historical source dealing with a piece of mathematics we know but with an approach that is foreign to us, we cannot dismiss it as "incorrect" in the same way that we as teachers may dismiss an unexpected student approach. Then, an effort may be required to make sense of it, and this "deciphering" requires exercising a similar kind of decentering and unpacking needed for listening to students. Thus, working with teachers on activities of reading and understanding idiosyncratic ways of doing mathematics that are typical in many historical sources is a way of learning to listen. We have tried such an activity with prospective teachers using an extract taken from the Rhind Papyrus that presents what today is called a linear equation with one unknown. A detailed description of the activity and the findings of the experience can be found in Arcavi & Isoda (2007). The activity of deciphering the primary source, even when it required much effort, was effective in supporting and nurturing the sensitivity towards and the capability of interpreting and understanding the other's perspective.

REVISITING WHAT WAS TAKEN FOR GRANTED

> "I have observed, not only with other people but also with myself […] that sources of insight can be clogged by automatism. One finally masters an activity so perfectly that the question of how and why is not asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question (Freudenthal, 1983, p. 469).

The development of ideas in history provide a repertoire of mathematical situations regarding such questions that, according to Freduenthal, "are not asked anymore". Consider, for example, the following text taken from a letter by the mathematician Antoine Arnauld (1612-1694) to a colleague, Jean Prestet (1648-1691).

> "…1 is to 3, as 4 is to 12. And 1 is to 1/3 as 1/4 is to 1/12. But I cannot adjust this to multiplications of two minus. For will we say that $+1$ is to -4, as -5 is to $+20$? I do not see it. Because $+1$ is more than -4. And conversely -5 is less than $+20$. Whereas in all other proportions, if the first term is greater than the second, the third must be greater than the fourth." Schrecker (1935, p. 85, translated from the original in French).

This text was presented to teachers in several workshops and they were asked to formulate a resolution of this contradiction (as if a student would have posed the question) between the idea of proportionality and the rule for multiplying two negative numbers. Working through such a task is a way of enhancing aspects of teachers' Specialized Content Knowledge.

Original texts as interlocutors

The Principles of Algebra by William Frend (1757-1841) was published in London in 1796. In this book, there is a virulent attack on the use of negative numbers. The following are extracts reflecting the argument.

> "The first error in teaching the principles of algebra is obvious on perusing a few pages only in the first part of Maclaurin's Algebra. Numbers are there divided into two sorts, positive and negative; and an attempt is made to explain the nature of negative numbers, by allusions to book-debts and other arts. Now, when a person cannot explain the principles of a science without reference to metaphor, the probability is, that he has never thought accurately upon the subject. A number may be greater or less than another number; it may be added to, taken from, multiplied into, and divided by another number; but in other respects it is very untractable: though the whole world

> should be destroyed, one will be one, and three will be three; and no art whatever can change their nature. You may put a mark before one, which it will obey: it submits to be taken away from another number greater than itself, but to attempt to take it away from a number less than itself is ridiculous. Yet this is attempted by algebraists, who talk of a number less than nothing, of multiplying a negative number into a negative number and thus producing a positive number, of a number being imaginary. …

This is all jargon, at which common sense recoils; but, from its having been once adopted, like many other figments, it finds the most strenuous supporters among those who love to take things upon trust, and hate the labour of a serious thought."

Frend's arguments can be summarized as follows:

- Rejection of "reference to metaphors" ("debts and other arts")
- Numbers only as magnitudes ("one will be one")
- Rejection of an extended version of operations (taking away only the small number from the greater, otherwise ridiculous)
- Rejection of an extended version of number ("a number being imaginary")

These claims may open up productive discussions about the nature of mathematics and mathematical objects, the place of generalizing beyond the concrete versus the useful role of concrete didactical resources/models for illustrating abstract ideas.

Frend's book has many mathematical developments which he juggles in order to avoid the use of negative numbers. Of special interest is his treatment of the solution of quadratic equations that purposefully avoids negative numbers. Not only are his mathematical arguments worth following, but they also provide a vivid experience of how the separation of cases (to avoid negative numbers) losses the efficiency, parsimony and elegance of generality.

FINAL REMARKS: A COMMENT AND A CALL

Since we are talking history, a brief comment on history of mathematics education is in place. Proposed roles of history of mathematics in mathematics education can be traced back to more than one hundred years ago. Jahnke et al. (2022) describe in detail the plea for the case of history of mathematics by three renowned mathematicians: Henri Poincaré (1854-1912), Felix Klein (1849-1925) and Hans Freudenthal (1905-1990), with special mention of teacher education. It is interesting to follow how their arguments evolved into the systematic work of many colleagues (see the case of HPM). Nowadays, the theorizing of teacher knowledge and teaching practices allows refining and substantiating those arguments, offering opportunities and frameworks for further theoretical and empirical work in the field. There may be many more examples for these roles, and many other possible roles to be explored and tried in settings of teacher education and teacher professional development. A possible way to advance these ideas would be to think how to bring closer the activities of PME and HPM.

As I said in the beginning, I am fully aware of the very burning issues mathematics education is facing, one of them is the theme of this conference. However, we need to keep making room for issues and directions of research and development which nowadays are not at the forefront of the agenda, like the topic of this presentation as well as others.

I conclude by borrowing the closing remarks of the opening address by Hans Freudenthal at the very first ICME in 1969 in Lyon:

> "On behalf of those who have worked hard to make this congress a success I welcome you and I invite you to use this week of scientific and social events as a great opportunity to exchange experiences and ideas, to meet people from nearby and far away, and to enjoy all good things this country and this city can offer you." Freudenthal (1969, p. 6).

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TEACHING IN THE NEW CLIMATIC REGIME: STEPS TO A SOCIO-ECOLOGY OF MATHEMATICS EDUCATION

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The phrase "socio-ecology" points to the need for combining socio-political and ecological concerns. In this plenary talk, I consider what the socio-ecological means for mathematics education and what responsibility mathematics education has towards the socio-ecological. I review my own past research on "what" we teach and "how" we teach it, in relation to socio-ecological concerns and propose eight themes. These themes include: questioning what gets centred in our work; moving towards a communal mathematics; engaging in a dialogic ethics; working against the epistemological "error" of focusing on the individual as the unit of learning. I conclude by calling for more work that investigates socio-ecological practices.

INTRODUCTION

We are in a new climatic regime (Latour, 2017); the patterns and relative predictability of the past have gone and will not return in our lifetimes. In this plenary, I want to consider, what does the new climatic regime mean for teaching and learning mathematics? The sub-title of this talk is a homage to the work of Gregory Bateson and is taken from his seminal book "Steps to an ecology of mind" (Bateson, 2000). I also want to acknowledge the influence of the following scholars on the ideas and practices I am going to describe: Laurinda Brown, Nathalie Sinclair, Dick Tahta, Tracy Helliwell, Armando Solares-Rojas, Kate le Roux (and there are many others). I will include specific references where I can and, while not placing on them any responsibility for gaps or prejudices in my own thought, their influence is throughout.

In using the term "socio-ecological", what I want to point to is that every concern about the planet's ecology has a socio-political aspect, and every socio-political concern has an ecological aspect (Coles, 2022). An analogy is the hyphen used in the concept "space-time". We may talk about space and about time, but since Einstein, we also know that, in our best models of the world, these two concepts are actually one.

It has been marked in several articles (Boylan & Coles, 2017; Barwell et al., 2022) that there has been an odd inattention, within mathematics education, to questions of the climate and the broader living world. We have researched the words spoken in mathematics classrooms, and not usually considered the quality of the air people breathe, to speak those words. The key theme of this plenary is to consider: *what does the socio-ecological mean for mathematics education and what responsibility does mathematics education have towards the socio-ecological?* (with thanks to

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 17-33). PME 46.

Mogens Niss for offering these paired challenges at a symposium on the socio-ecological).

Wolfe (2020) proposes attention to the "what" and the "how" of what we do and think, and I use this distinction to structure a looking back at my research, as it relates to the socio-ecological. In listening to this talk, no doubt your attention will shift between "what" I say and "how" I say it. (I suspect the "how" may actually stay with you, or strike you, more strongly than the "what".)

The sections of this report are as follows. Firstly, I consider possibilities for "what" we teach, if we think of the socio-ecological together. Secondly, I consider possibilities for "how" we teach, again, thinking of the socio-ecological as denoting non-separate domains. Looking back over these two sections, I then set out themes I observe in a socio-ecological practice of mathematics education. I close by looking forward to future possibilities, bringing together the what and the how of teaching.

RESEARCHING THE "WHAT" OF TEACHING

The "what" of teaching, I take to be concerned with curriculum innovation and the content of what goes on in schools and universities, in the teaching of mathematics, or the teaching of teachers of mathematics. In this section, I set out my own learning from a selection of projects I have been involved with (in England and Mexico), each linking to a curriculum innovation that has some connection to the living world; the section addresses what the socio-ecological can mean for mathematics education. My own work has largely been practitioner research and small scale. The first PME research report I wrote was co-authored with Laurinda Brown in 1997 (Brown & Coles, 1997) and, until 2010, I was working full-time in a secondary school. Since 2010, I have been at the University of Bristol and have been involved in larger projects (some of which I will describe) but practitioner research remains a key part to my work. Accounts, below, are offered largely descriptively as, in some sense, these experiences provide the "data" which has led to the later theorisation of the socio-ecological.

Teaching mathematics as if the planet matters

In 2013, Richard Barwell, Laurinda Brown, Tony Cotton, Jan Winter and I collaborated on a book project that was part of a series on "Teaching school subjects as if the planet matters" (Coles et al. 2013). The series was halted after the first few subjects, I assume through lack of interest or uptake. Our book was aimed at practitioners and was structured in two parts. The first tackled five socio-ecological issues (including food, biodiversity and the economy) and showed how these issues could inform mathematics teaching (i.e. how lessons could begin with these issues and draw out elements of the curriculum). The second half tackled elements of the English curriculum (including algebra, number and statistics) and offered tasks which addressed those areas, while drawing on contexts relevant to the living world.

One of my contributions was a chapter on biodiversity (Coles, 2013). I researched the collapse of the cod fish stocks off Newfoundland. One of the things of interest that I found was the way in which estimates of the size and health of the fish stocks were modelled mathematically. In analysis done after the dramatic collapse of stocks, it became clear that political interference in the mathematical modelling had led to overestimates of the health of the fish numbers and hence the acceptance of over-fishing. It appears as though politically expedient decisions (to allow more fishing) were masked by the seeming objectivity of a mathematical model. Critical mathematics education (Skovsmose, 1994) points to the need for a reflective knowing about mathematics, i.e. to question how models are used, for what purpose and to whose gain. In this case, mathematical modelling seems to have contributed to ecological collapse.

A symposium leading to a special issue

Following publication of the book described above and encouraged by the work of Barwell (2013) and work on climate change and mathematics education in Norway, I collaborated with Mark Boylan on a symposium for the British Educational Research Association (BERA) in 2016, on mathematics education and the living world. This led to an international call for papers for a special issue of the Philosophy of Mathematics Education Journal. In 2017, the special issue “Mathematics Education and the Living World” was published (see, Boylan & Coles, 2017). Looking back, it is clear the socio-political and ecological are entangled throughout the writing in the special issue, for instance, in the mixing of themes of sustainability and social justice.

One highly cited paper in the special issue is Gutiérrez (2017), centring Indigenous knowledges and introducing the term mathematx, to propose a new form of mathematics, marked by reciprocity, responsibility to others and by joy, again mixing the political and the ecological. My own contribution (Coles, 2017) took up the question of habit, and what habits might be needed in the Anthropocene (a term I tend not to use now). In particular, I explored Bateson (2000)’s proposal for a recursive logic – I return to Bateson’s ideas in discussion of the socio-ecological.

School curriculum innovation: the Atoyac River (Mexico)

At the same time as the special issue, I began a research collaboration with Armando Solares, which continues to this day. We initially won a British Academy “Newton Mobility Grant” to explore: “Connecting out-of-school experience with classroom mathematics in culturally diverse societies”. Armando, one of whose backgrounds is in mathematical modelling, wanted to investigate the experiences of primary children who are in paid work, often on the street, and often meaning they have to miss school.

Following this project, we were then successful in a bid to the UK’s Research Institute as part of their “Global Challenges Research Fund” (GCRF) (project title: “Community, Science and Education: An interdisciplinary perspective for facing environmental crises in Mexico and South America”). Through this GCRF project, I have been privileged to work with a group of scientists, teacher educators, teachers and community activists, on the challenge of innovating the curriculum in a school that is sited next to a highly polluted river (the River Atoyac). I spoke about this project last year at PME (Coles et al., 2022) and so will only touch on it here briefly.

The GCRF work drew on ideas of socio-critical modelling (Solares et al., 2022). It was affected by Covid, and extended periods of Mexican school closures. Nonetheless, the network was able to bring to fruition a year-long curriculum innovation programme in a primary school in Mexico which culminated in a “memorial museum” to the Atoyac. This museum had three galleries, one looking to the past in which children gathered some of the oral history of the river and its significance to the community that lives in its basin. This gallery was seen as significant, since the Atoyac has been polluted for so long that primary age children have not known anything different.

A second gallery focused on the present and the current state of pollution. We had imagined that children might be able to collect data themselves from the river, to track it pollution levels. The scientists in the network advised that the river is so polluted that this would be dangerous. Nonetheless, data was gathered and used in this gallery. A third gallery focused on the future and what might be required to move make to a healthier state for the river. The future gallery aimed to support activism.

A website offers a virtual museum (https://red-comunidadcienciaeducacion.org) and the community members of the network are taking a physical version to over 20 sites in the region. One difficulty that had to be overcome was the centralised Mexican national curriculum, which is delivered through national text-books. Teachers needed to find connections between the museum project and what they have to teach via text-books. Although some links to mathematics were made (Solares et al., 2022) it is also noticeable that there were potentially more opportunities which were not taken up.

Teacher education curriculum innovation: the Green Apple project (England)

I have been involved in an on-going project with prospective teachers. Since 2015, there has been an offer to prospective teachers at the University of Bristol to join a “Green Apple” project. The name came from a fund that initially offered money to support group meetings (the money allowed us to buy pizza). The aim of the project is to encourage and support prospective teachers to bring issues arising from global challenges into their own teaching. A recurrent theme is how to make sense of global issues on a local scale and, also, how to link local concerns to wider/global challenges. One finding has been the use of intermediate scales to help link local and global, for instance, considering the area of a school site, to help conceptualise deforestation rates.

With Tracy Helliwell, (Coles & Helliwell, 2023) we have analysed this on-going work and, in particular, the different roles of the mathematics teacher educator (MTE). We have proposed the following roles for MTEs wanting to support prospective teachers drawing the living world into their classrooms: supporting prospective teachers in identifying "generative themes", i.e., issues of relevance to themselves, their students and the school community; questioning the ways in which mathematics is portrayed in classrooms, i.e., what functions or forms of mathematical knowing are allowed; questioning assumptions made by the prospective teachers, which may not be noticed.

In Coles and Helliwell (2023), we recognise that the tasks developed by prospective teachers, with a socio-ecological element, are likely to be one offs in their practice. We also reflect that our focus on socio-ecological issues, in teaching prospective teachers, is not overly prominent and seems far from a consistent theme, partly as a result of the constraints of a national curriculum for teacher education.

Teach the future

I was privileged to be invited to do some work for a UK-based student-led campaign group "Teach the Future" on a re-envisioning of the mathematics National Curriculum in England, for 11-16 year olds. The outcome of this work was to propose a new set of distinctions for organising the curriculum, replacing "Number", "Algebra", "Geometry", and so on. Rather, the proposed focus is on a mathematics for sustainability, organised around the labels: Measuring, Changing, Mapping, Risking, Deciding (adapted from Roe et al. 2018). Each category came with a core question:

- Measuring: how big (or small) can things get?
- Changing: how quickly do things change?
- Mapping: how do things occupy space and time?
- Risking: when is a risk worth taking?
- Deciding: how do we act ethically?

These proposals are part of a campaign to get the UK government to consider curriculum innovation. Could sustainability be placed at the centre of a mathematics curriculum? There are also going to be linked to resources for teachers to use, in the context of the current curriculum organisation.

Themes from the "what" of teaching

Looking back over almost a decade of research, one observation is that the work remains at the margins, in relation to the concerns mandated by the governments of the countries in which I have worked. The work, nonetheless, feels important, in terms of exploring possibilities for task design. For instance, the framing of a "memorial" or "memory" Museum, which looks to the past, present and future, is one that potentially could be applied in many contexts of socio-ecological degradation or pollution.

For both teachers and students, bringing the living world explicitly into the classroom can provoke strong reactions, including grief and a sense of loss or powerlessness. These are emotions that need to be managed and space given for their expression. There is a need to consider how to balance more dystopian projections for the future, with hope (e.g. from the way the world collaborated on reducing Ozone). In some of the work, there is a strong sense of community and learning mathematics not as an individual endeavour. The two themes, of a communal mathematics and the raising of questions of affect and the unconscious, are ones I take into the next section, which moves on to the "how" of teaching. The "how" of teaching begins to touch on the responsibility of mathematics education towards socio-ecological challenges.

RESEARCHING THE "HOW" OF TEACHING

In contrast to the difficulties of changing the "what" of teaching, which can require wide ranging system change, the "how" of teaching can often be altered much more readily, potentially impacting practices of teaching, and not just at the margins. This is not to say changing how we teach is easy, but rather there may be fewer constraints.

This section, focused on the "how" of teaching, of necessity will move away from a direct consideration of socio-ecological issues. However, I believe the "how" of teaching may be the most important site for development of a socio-ecological practice. What I focus on in this section is research which, whether explicitly or not, moves towards a sense of learning as a community and pays attention to affect. Within a communal mathematics there is the possibility of recognising others' contributions to shared aims. Others (human and non-human) can support possibilities for new understandings of the living world, "seeing-as" (Zwicky, 2003, §3) in novel ways.

I believe I first heard the phrase "communal mathematics" from conversation with Dick Tahta. I was fortunate to meet Dick and know him for the last 10 years of his life (Dick died in 2006). Dick was born in Armenia and was an inspirational figure for many mathematics teachers in the UK and beyond, founding the "Leapfrogs" group who developed resources for the classroom. Dick's writing offers a vision of a way of teaching that I do not see as being recognised in research. It is not discovery learning (although there are elements of it) and the teaching is not based on memorising algorithms (although there are elements of that too). Dick was an influence on Laurinda Brown (with whom I have worked over many years). Two of the most powerful experiences I have had, in mathematics education, were sitting at the back of a classroom and watching Laurinda and, years later, Dick teach one of my classes. I have spent over 25 years trying to understand what is so compelling about their ways of working (which is not to say they teach in the same way). In the next sub-sections, I describe my on-going attempts to articulate this understanding, focusing on elements of a communal mathematics and the recognition of the role of affect in learning.

Developing a need for algebra

In 1998, in one of the first research grants I won, I collaborated with Laurinda Brown to work on the question of how to teach students aged 11-12 in such a way that they developed a "need" for the algebra they had to learn. The project fund's purpose was to encourage practitioner-research, supported by a University educator. Our work on this project provided a template of how Laurinda and I would continue to work for many years, with co-planning of lessons, Laurinda coming in to observe and sometimes co-teach, and then reflection afterwards. There were striking examples of students using and "needing" algebra on the project. As one outcome, we articulated four practices which seemed significant in students developing a need for algebra:

- developing a wider classroom culture to do with 'becoming a mathematician';
- the teacher commenting on and highlighting the mathematical behaviour of students whenever it was observed;
- the choice of activities and teaching strategies used within those activities e.g. the use of common boards;
- an emphasis on students writing both in the act of doing mathematics and in reflection on what they have learnt (Coles, 1999).

These four practices were an attempt to capture teaching strategies which seemed to support a classroom in which there could be deliberate decisions by the teacher about specific content to be learnt and at the same time, choice and autonomy for students.

I still have some students' exercise books from 1998/9 and have recently collaborated on a new analysis (Coles & Ahn, 2022). The third strategy, "use of common boards", is directly linked to working *communally*. On a common board, students might pin up work, or write up results, such that each individual's work was available to the whole class, e.g. to support spotting of patterns. The framing of the project (finding a "need" for algebra) speaks to tapping *affective* responses of engagement and even excitement.

Symbolically structured environments

Many years later, as part of an on-going collaboration with Nathalie Sinclair (Coles & Sinclair, 2019), we wrote about a commonality we observed across the teaching of Bob Davis, work I was doing using Gattegno's "Tens Chart" and work using Jackiw and Sinclair's "TouchCounts" iPad app. A breakthrough insight came from studying a remarkable video of the teaching of Bob Davis. In this video, Davis gets young students working confidently with negative numbers, through a task in which pebbles are put into and taken out of a bag which contains an undisclosed number of pebbles. He records on a board the number put in, taken out and the change in stones, compared to the start. So, a statement such as $3 - 3 = 0$, means, in the context of this problem: I have put 3 stones in the bag, taken 3 stones out of the bag, and there is no change compared to when I started. In the video, we observe the students asking Bob Davis to put 5 stones in and take 6 out, leading to: 5 – 6 = -1, indicating there is one less stone in the bag now, compared to the start. Negative numbers appear to arise in a natural and unproblematic manner, for children who look around 6 years old.

It is an astounding video clip. They key, for me, is that when Davis writes "3" or "6" on the board, what he is not doing is referring to stones. What these symbols represent are actions on the stones, or changes in the number of stones. And with this subtle shift (from stones, to actions on stones), the world of negatives opens up. Rather than the ontological conundrum of "negative stones", children instead can draw on deeply embodied senses of how actions can be done and undone. I can put in and take out, and by symbolising this action, there seems no difficulty in symbolising the situation where I take out more than I put in. Nathalie and I (Coles & Sinclair, 2019) distinguished these features of what we came to call symbolically structured environments (SSEs):

(a) symbols are offered to stand for actions or distinctions;
(b) symbol use is governed by mathematical rules or constraints embedded in the structuring of environment;
(c) operations can be immediately linked to their inverse;
(d) complexity can be constrained, while still engaging with a mathematically integral, whole environment;
(e) novel symbolic moves can be made.

The video clip, as described, offers one example of (a), (b) and (c) and how these might link to a *communal mathematics*. Symbolising actions and distinctions is a task that requires communication, it requires others. Constraining symbol use with rules can be done powerfully in a game like situation, where *affect and cognition* are linked. Our experience is that students can thrive on being placed in a structured "whole" mathematical situation (d), and become engaged in the possibilities for novelty (e).

Differentiation from an advanced standpoint

I was co-investigator on a project, at the University of Bristol and led by Ros Sutherland, aiming at "Widening Participation" in the University, i.e., making routes to study at Bristol more accessible for those people who have suffered most disadvantage in their lives. The project centred around supporting 20 teachers (in two groups of 10) to engage in practitioner research in their classrooms, using an action research framing. In the first cohort of teachers, an idea emerged from one teacher, which spread to half of the group, of approaching widening participation by choosing to teach students aged 15-16, who have been failing in mathematics relative to their peers, topics which would be deemed several years "ahead" of where they are at in any kind of linear curriculum map (e.g. offering Pythagoras' Theorem and trigonometry to students who would never have usually been exposed to such ideas).

The results were striking. In all cases, the teachers reported students, within a short space of time, beginning to speak quite differently about themselves and mathematics, e.g. "I'm doing A grade work". In internal test scores, the students, on average, made significant progress in a short space of time. Some time after the project ended, Laurinda Brown and I (Coles & Brown, 2021) coined the phrase "differentiation from an advanced standpoint" to try to capture what these teachers had done. These teachers were not concerned that all elements of prior knowledge have to be secure before moving to a new topic (see Coles & Sinclair, 2022). Instead, they aimed at a complex idea, potentially considered to be several years in advance of students' current work and differentiated (i.e. made accessible) from that advanced standpoint. The teachers' raised ambitions were associated with a shift in their students' *affective* relationship to mathematics and some teachers also reported their students becoming more collaborative (*communal*) in their approach to learning. I find it a source of hope that students, even at age 15-16, can seemingly change their view of the subject so readily.

Themes from the "how" of teaching

Looking back over this work, which took place over a 25-year period, I would like to characterise, further, what I see as similar across the examples and articulations, about an approach to teaching that I believe has no commonly recognised label.

A commonality is that mathematical concepts are offered as actions, relations or transformations. Klein (1893), as part of his Erlangen programme, conceived of geometry as centred around transformations. The video of Davis teaching negative numbers offers an example of how a transformation approach could be applied to other parts of the curriculum (see Norton, 2019 for an article on extending the Erlangen programme). Focusing on action can make mathematical symbolism tangible and this is, from my experience, a key element of engaging students affectively.

In all cases (above) teachers are working with students on content that would typically be judged as two or more years "above" any age-related curriculum plan. This is perhaps the simplest and maybe even the most profound distillation of what is going on and what, as a teacher, you might be interested to try out yourself. By setting up the context of teaching several years ahead of what students need to actually master (according to the curriculum) a certain freedom is introduced. If designed carefully, a teacher can introduce a task which allows genuine choice and autonomy for students, knowing that whatever line of inquiry students take, they will be practicing those skills which are on the curriculum for their age. This idea perhaps needs illustration.

One example of melding a space for creativity with learning curriculum content, from my own practice is school, is a project we called "Graphs of Rational Functions", where the challenge for students was to predict the shape of the graph from equations of the form: $y = \frac{ax+b}{cx+d}$, (and potentially extending to quadratics in the denominator). This was a topic for grade 13 students at the time; we would use it with grade 10s.

In this task, there is a process (close to an algorithm) which students need to follow, to create a sketch of a rational function. We would work together on what to do (e.g. finding y when x is zero, finding x when y is zero, checking for large values of x, etc). Having set up this process, students can explore any set of rational functions they like. As a teacher, I need have no agenda about what might or might not be discovered about rational functions, since this was not something students needed to know in grade 10. What I was observing and noticing was that, in drawing graphs of rational functions, students were becoming skilled in substituting into formulae, solving equations, plotting co-ordinates, estimation, and much more.

Working well ahead of any mandated curriculum is also a key to working communally. If there is no urgency about students' understanding of, say, graphs of rational functions, then it may be easier for space to open for collaborative ways working. My attention as a teacher need not be so focused on who has done what, as individuals, but rather on where we have got to as a group. And working on potentially complex tasks introduces a reason why working together may be necessary to arrive at solutions.

The theme of affect links to the unconscious psychological processes at play in any classroom setting. Tahta (1993) is one of the few mathematics educators to consider insights from psycho-analysis. What cannot be captured, in the descriptions above, are the inter-personal forces at play. In my own PhD, I became interested in issues of how teachers listen (Davis, 1996) and the effect that an evaluative listening (where student comments are judged right/wrong) has on closing down spaces for dialogue. There is not space to consider psycho-dynamic links in detail here, except to say that, in tackling socio-ecological questions, these may become particularly pressing issues.

Having set out ideas around the "what" and the "how" I now draw on that work to think in a more philosophical manner about a socio-ecological practice of mathematics education. Following this theoretical interlude, I end with some thoughts on the future.

A SOCIO-ECOLOGICAL PRACTICE

My aim in this section is to re-look at the "what" and "how" of teaching and offer an analysis of how practices relate to the socio-ecological. I view a socio-ecology of mathematics education as a process, or practice and, at present, interim implications are being worked out collectively and through dialogue. The processes and concerns I associate with the socio-ecological share commonalities with several other approaches to research, e.g. ethnomathematics, critical mathematics education, work on Indigenous ways of knowing and socio-political research (see Coles et al., 2022). The socio-ecological is not, therefore, a unified "perspective", or an approach that aims to reject others, rather it is a conjunction, that *and* this.

The phrase socioecology has been used in the past, to mean something different. Notably, Bronfenbrenner (1977) proposed a socioecological psychology (an idea which has been picked up recently, Louie & Zhan, 2022). In Bronfenbrenner's work (and other related studies) the socioecological is conceptualised as a set of concentric circles emanating from the individual. These circles (microsystem, mesosystem, exosystem and macrosystem) offer an image of how individuals are embedded in the world and influenced by factors operating at vastly different scales. My own use of the phrase socio-ecological is slightly different. Whereas Bronfenbrenner has the individual at the centre, (along with others, Coles et al., 2022) I want to work to de-centre the human. The image of the human as locus of attention and concern can all too easily lead to a view of what is outside the human as existing solely for the benefit of (some) humans. When I speak of socio-ecological practices, I am wanting to evoke practices which displace the intact, isolated human and draw attention to relationships.

In a beautiful book that re-imagines our relationships in the world, Aït-Touati et al. (2022) offer an alternative image of what they refer to as "the point of life", to represent life from an "animate body, a living point" (p.55). Rather than a God's eye view, with individuals separated from milieu, Aït-Touati et al. play with the mathematical function of inversion, to represent the point of life as a circle, edged by skin, where inside the circle, the rest of the world is reflected in envelopes of increasing scale. Inside and outside are no longer separate domains. The skin acts as a boundary and a link. The point of life is not closed in on itself and has not internalised the world, instead, "the interior world is equivalent to the world itself" (p.60) in the same way any fragment of a hologram contains an image of the whole. Aït-Touati et al. (2022) argue, "there is no physical milieu, because a living thing's milieu is other living things and is the product of their metabolic activity in the past and present" (p.62). When I use the words "ecology" or "nature" I am wanting to invoke just this kind of image of an inter-leaved, combined, symbiotic relationship between living beings and their milieu.

In de Freitas et al. (2022) a group of mathematics educators explore possibilities for using diagramming practices, and functions such as inversion, to help work on processes of creative abstraction. This article linked to a World Universities Network, co-led by Kate le Roux and myself, taking place across 7 countries (https://wun.ac.uk/wun/research/view/innovating-the-mathematics-curriculum/), on (socio-ecological) curriculum innovation at Higher Education. I now look back over the research I have presented, to point to eight strands, relevant to a socio-ecology of mathematics education; I offer the strands in two groups, responding to the paired questions posed at the start of this report.

The implications of the socio-ecological for mathematics education

1) At the heart of a socio-ecological practice is refusing to take the living world as a fixed background for social or socio-political concerns, and refusing to take the socio-political context as a fixed background for ecological concerns. And this leads directly to *questioning what gets centred in our teaching and research.* In the River Atoyac work, in Mexico, the human subjects were not the prime focus, neither was the mathematics. Instead, it was the river which was at the centre; the outcomes of the project were primarily about how the river could come back to health. One lesson from the River Atoyac work, is the critical role of engagement of the community partners.

2) Equally important, is not taking the socio-political as fixed background of ecological concerns. For example, when bringing contexts of climate change into the mathematics classroom, *considering issues of climate justice* are unavoidable (e.g. considering the historical processes that result in asymmetric distributions of vulnerabilities and risks).

3) There is a pull away from a focus on individuals (something discussed further, below). In the projects described there is a sense of *a communal mathematics*. Students are invited to work together, to share their results on common boards, to notice each other and build on each others' thinking, skills central to a socio-ecological practice.

4) There is *attention to affect and cognition*, not viewing these are separate processes. In particular, there is attention to the ways in which students (and teachers) may experience climate grief or despair, and also need to find hope and community.

The responsibility of mathematics education towards the socio-ecological

5) Ethics are central to a socio-ecological practice. In Barwell et al. (2022) we proposed a *dialogic ethics* as a suitable approach to our relationships in the living world, in the context of teaching and learning mathematics. A dialogic ethics begins with the "other" and the sense that it is through relationship with an other that I come to recognise my responsibility for others. The other summons us to responsibility and, in being summoned, we can recognise our unique answerability to those others. A MTE, or teacher, can act as an "other" for their students, in supporting those students to recognise their own uniqueness and responsibility. Classroom practices that support students in asking their own mathematical questions and following their own lines of inquiry offer one image of what it could be like to enact a dialogic ethics.

6) The very phrase socio-ecology points to a complex and entangled relationship between a self and an other, or between system and environment, or nature and culture, or inside and outside. Rather than separate, a socio-ecological practice conceptualises culture and nature as one realm. In *thinking nature-culture, or, the socio-political and ecological together,* we are forced to contemplate the many

varieties of culture and intelligence across human and non-human life. The idea of "differentiation from an advanced standpoint" also challenges perhaps typical Western notions of intelligence and the capabilities of students in school, in opening up spaces for new relations.

7) Bateson (2000) highlighted the epistemological error in Western life, which is a focus on individuals as the unit of survival, evolution and learning (Wilden, 2001). It is this epistemological error which has led to a disposal society, though trying to maximise profit and growth. And yet, it is clear that a system which disposes of its environment, disposes of itself (Wilden, 2001). Instead, Bateson proposes, we need to focus on *relations as the unit of survival, evolution, learning*. In an age of relativism, it might seem odd to talk about an "error" in relation to epistemology. For Bateson, the word error points to the idea that our theories about the world need to respect the complexity of the world. Moves towards a communal mathematics, enacted in symbolically structured environments, offer a mechanism for a focus on relations, both within the mathematics and among the people doing mathematics.

8) One aspect of the epistemological error of the focus on individuals is a focus on the identity relation, i.e. an interest in application of the logical statement A=A, a search for when two seemingly different things are the same. Korzybski (1993), in a book which influenced Bateson, proposed a paradigm which had at its starting point the insight that there is no identity relation. *Nothing is identical to another thing*. Instead, the fundamental relation is one of difference or differentiation. I might conceptualise learning as an event, or a process, which passes through me, leaving a residue, which I mistake as its essence. If I leave aside that there is any such thing as identity (Korzybski, 1993), perhaps it is easier not to get caught in identifying individuals (myself, my students) with individual attributes and coherent identities. The events I mark as an "understanding" or a new "learning" are parts of wider processes, extending beyond my skin to partners, tools and all that sustains them – the unit of learning is not the individual. As Dick Tahta pointed out, as soon as I start thinking about individual understanding, I am led to conceptualising *who* understands, and to categorising those who seem to never understand. Yet, I always have some understanding and will certainly never have your understanding.

LOOKING FORWARD

The practices I have offered, prompted by what the socio-ecological means for mathematics education, are: *questioning what gets centred in our teaching and research; considering issues of climate justice; a communal mathematics; attending to affect and cognition*. And, in terms of what mathematics education could have to offer in the face of socio-ecological challenges, I have considered: *a dialogic ethics; thinking the socio-political and ecological together; relations as the unit of survival, evolution, learning; nothing is identical to another thing*. This second group of practices point to the need for encounters in mathematics education to embody alternative ethical, political and aesthetic relationships, as Paola Valero's plenary talk argues powerfully.

I aim now to draw together the strands above, to consider what a classroom might look like that combined them. And I experience a tension, between wanting to offer something relatively utopian and wanting to offer an image that feels do-able today. Perhaps both are needed. If we threw everything in the air and asked how we wanted to organise schools in the new climatic regime, I imagine these might be schools closely linked to their communities and the concerns of those communities, while also with enough distance to question, challenge and critique those communities. I imagine these might be schools where students are regularly outside (when that is safe) and where the questions which are dealt with, on the curriculum, arise through dialogue between students and between students and teachers. We do have examples of such schools and universities (the Reggio Emilia pre-schools in Italy, the Green Free School in Copenhagen, the Green School in Bali, Black Mountain College in Wales).

Perhaps a closer-by image is also useful to consider. What might be possible to do in a curriculum still structured according to subjects such as "mathematics"? The example, earlier, of graphs of rational functions has a structure of a beginning in which students are shown (in a relatively constrained manner) how to generate these kinds of graphs, before being given space to try out their own versions and attend to patterns in what they find, all the while practicing key mathematical skills on the curriculum. How might such a task link to explicitly socio-ecological issues? Looking at the Teach the Future curriculum, the most obvious connection is with: "Changing: how quickly do things change?" and its links to socio-ecological questions.

I offer a brief imaginative account of a task progression. I might start by asking "What things change" and making a list of student responses, before prompting, "And how do these things change over time?". I could imagine gathering sketches of graphs, with time on the x-axis. There might be an opportunity for students to raise their own questions. All this, in a sense, is to generate a "need" for studying ways in which things can change over time, e.g. for studying the graphs of functions. Having generated a need, a task design similar to graphs of rational functions could be enacted (perhaps, for younger students, focused on relationships of the form $y=mx+c$). Such work could continue for several weeks. Having gained some skills of drawing and predicting graphs from equations, students might then return to the questions at the start, of what changes and how do they change. The teacher might have prepared some data, relating to the socio-ecological issues raised by the students in the first lesson, perhaps in the form of graphs. The task for students might be to find equations which match the graphs, or parts of the graphs, and use those equations to make predictions of the future.

For instance, students might have a graph of global CO_2 levels, over the last 10 years, and be asked to predict future levels … And what if mathematics teaching did not stop there? What are the mechanisms for surfacing affective responses, while maintaining the emotional safety of the classroom? What are the mechanisms for discussing the climate justice aspects of rising CO_2, and in a manner which is honest? What teacher education is required? And, what are the possibilities for schooling to become a site of activism? My colleague Tracy Helliwell has been exploring the potential for arts-based practices in addressing climate justice with teachers. I have an on-going collaboration with Penny Hay which, last year, led to a "Living Tree Mirror Maze" project, sited in a theatre (which won a national award for inspiring future generations), aiming to combine mathematical awarenesses with reflections on relationships in nature (https://houseofimagination.org/foi-case-study-living-tree-mirror-maze/). There is much work needed, to explore and learn from such synergies.

In Boylan and Coles (2017) we offered a "map" of steps to developing a mathematics education for the living world, some of which have been taken; for instance, ICMI holding a symposium on "Mathematics Education and the Socio-Ecological" in March 2023, and a number of small-scale projects being conducted around the world, which are linking mathematics teaching to the living world. Some of the further steps we proposed were: schools experiment with a curriculum in which mathematical development is balanced with interdisciplinary work; teacher training courses shift to include, how the study of mathematics can relate to wider ecological issues and the living world; scholars from across the globe document, trial and share experiences of linking mathematics education to the living world. I hope to have shown that this is important, urgent work to try and that socio-ecological issues are relevant for the practices of mathematics education, as well as mathematics education having responsibilities, in response to our collective socio-ecological needs.

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IMPLEMENTATION OF COLLABORATIVE PROBLEM SOLVING: EXPERIENCES IN CHILE

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The 2030 Agenda for Sustainable Development, adopted by the United Nations in 2015, sets 17 goals for peace and prosperity for people and the planet. The fourth goal is to ensure inclusive and equitable quality education and promote lifelong learning opportunities for all. In many countries, including Chile, quality education is not available for all mainly because of socioeconomic exclusion, but also because of other exclusion factors taking place in classroom. Mathematics is part of a quality education and to make it available for all students is an important step for reaching the fourth goal. In this article, I present a classroom proposal for students' mathematics learning and abilities development. We then present a proposal for professional development for teachers and for facilitators as means for scaling up the classroom proposal. I complete the paper presenting three implementation cases where difficulties, achievements and complexities of this endeavor are analized.

INTRODUCTION

The ultimate goal of mathematics education research is to improve mathematics teaching and learning at all levels. Along the years, ample knowledge has been developed, but most of it has not reached classrooms, teachers, and students. In the last years, however, an increasing interest in the implementation of research advances has taken place among researchers.

In this article, I describe the experience of implementation of collaborative problem solving (CPS) in Chilean classrooms. This experience has been developed as an independent program inside the University of Chile, under the umbrella of the Center for Advanced Research in Education (CIAE) and the Center for Mathematical Modeling (CMM). The program is identified as *ARPA Initiative* (ARPA is the Spanish acronym for Activating Problem Solving in Classrooms) and it is characterized by a step-by-step construction, with intermittent research and development funding. The construction steps have been defined by an internal implementation-evaluation cyclic process, a sort of *look-and-feel* (Koichu et al., 2022), to gain understanding on the materials, procedures, and learning experiences. The original motivation for ARPA Initiative was the inadequate achievements of Chilean students in mathematics (OECD, 2019) and the conviction that CPS is a tool for improving mathematics learning. ARPA Initiative takes roots on the actual work of mathematicians and experiences of PS in university and elementary classrooms. Our program is coherent with the National Curriculum (Ministry of Education, 2023) and it is influenced by the 21st Century Skills educational movement (Pellegrino et al., 2012). On the other hand, research by Garet et al. (2001), Desimone et al. (2002), Koellner et al. (2007), Stein et al.

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(2008), Marrongelle et al. (2013) Liljedhal (2014) and Nevado et al. (2015) have been behind many decisions taken along the way.

In this presentation, I have as a background three initiatives: the creation of the journal *Implementation and Replication Studies in Mathematics Education* (IRME), the work by the *German Center for Mathematics Teacher Education* (DZLM), and the analytical framework for problem solving (PS) implementation by Koichu et al. (2022).

The creation of IRME has been the result of an extended discussion at different mathematics educational forums. A well-documented description of the first attempts to define implementation in mathematics education is presented by Koichu et al. (2021). As part of this discussion, we must mention the publication of a literature review and 15 research articles in ZMD-Mathematics Education (Vol. 53, issue 5). In the opening article of IRME, Jankvist et al. (2021) further considered the academic interchange leading to the journal. At this point, the journal is established as a forum for a fundamental research area: implementation in mathematics education.

In the article by Roesken-Winter et al. (2021), the authors report on extensive research on the implementation of research-based innovations developed at the German Center for Mathematics Teacher Education (DZLM). Through a long-term research, design, and implementation iteration process (Penuel et al., 2011) a Three-Tetrahedron Model for professional development (PD) research and design was created at DZLM (Prediger et al., 2019). This model depicts three implementation strategies: a material, a personnel, and a systemic strategy, for three levels of formation: students, teachers and facilitators. Each of these levels is considered inside its environment: school, PD context, and PD facilitator context (Roesken-Winter et al., 2021; Prediger et al., 2019).

In a recent article, Koichu et al. (2022) proposed an analytical framework for the implementation of PS in classrooms. The authors introduced the notion of PS *implementation chain* as a dynamic sequence of intended, planned, enacted, and experienced activities shaped by researchers, PD facilitators, teachers, and students, respectively. Each of these four elements are connected through interactions among the element agents. On one direction, interactions among the agents on the PS resource and, on the other direction, reflective feedback of the agents for improving the resource and the enactment of the activity (Figure 1, in Koichu et al., 2022). The PS implementation chain defines a cycle that is repeated various times improving the elements of the chain and the interaction between the elements, resulting in a sort of *cycle of improvement*.

At this point, I want to thank Josefa Perdomo-Díaz and Cristian Reyes as founders of ARPA Initiative (Felmer el al., 2019) and all members of ARPA team. I also want to thank ANID Basal Funds for Centers of Excellence FB0003 and FB210005 for their continuous support.

WHY COLLABORATIVE PROBLEM SOLVING

The major goal of ARPA Initiative is to introduce CPS in mathematics classrooms in a context where students' achievements in mathematics are stagnant and traditional teaching is the common methodological approach.

Why problem solving? PS is internationally recognized as a mathematics activity offering students opportunities for mathematical development: connecting concepts, elements, and ideas, promoting skills such as examining, representing, and applying, and the use of mathematical thinking such as abstracting, analysing, guessing, generalizing, or synthesizing (Schoenfeld, 1985). The introduction of PS in classrooms has the potential of impacting the overall quality of mathematics learning.

Behind unsatisfactory mathematics results of Chilean students, there is a vertical classroom practice characterized by a passive role of students and mathematics activities with low cognitive demand (Radovic and Preiss, 2011; Rodríguez et al., 2013). In a more recent study, Saadati et al. (2018) examined teacher ability to reflect and make decisions on student work, showing that 62% of teachers did not consider (or did not realized) students' mistakes in their feedback. In another study, Donoso et al. (2020) observed classroom videos, when students solved problems, finding activities with low cognitive complexity, with no major teacher feedback and a predominant discourse of teachers.

Why collaborative? CPS has been considered by the ARPA Initiative, based on the premise that student collaboration would promote a change of mathematics classroom climate and would enrich teacher-student and student-student interaction. This basic idea was reinforced by the educational trend of 21st Century Skills where, as mentioned, collaboration and problem solving are considered basic skills to be developed. While having a CPS mathematics experience, students develop oral communication, active listening, decision making, leadership, reasoning, and critical thinking (Szabo et al.2020).

From a theoretical point of view, Koichu (2018) suggested that collaboration is one of the PS enhancers of ideas to solve a problem. Relying on the premise that key solution ideas for solving a problem are constructed as a result of shifts of attention, Koichu asserts that they are more likely to occur in learning environments with many resource options, and collaboration with peers is one of these options from which the solvers can choose. Pruner and Liljedahl (2021) analyzed the Thinking Classrooms of Liljedahl (2020) as an example of such a choice-affluent environment, where collaboration is one of its main options.

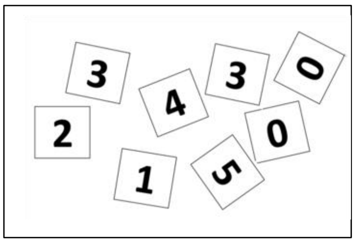
To conclude, I mention that PISA 2015 reported that Chilean students scored below the OECD average in the CPS test (OECD, 2017). However, PISA 2015 also reported that Chilean students like to collaborate (93%) and they believe that collaboration is good for reaching their goals (81%), both figures above the OECD average. This may be seen as a good starting point for developing CPS among students in Chile.

In the following four sections, we present our program for the implementation of CPS in classrooms. In our description, we use often use the term *proposal* rather than *plan* or *strategy* to stress that implementation of an innovation requires an adaptation process from the beginning and for a long time, according to school's needs, context, and unexpected circumstances. In what follows, we present our Classroom Proposal, then the first year of CPS Implementation Proposal, followed by Facilitators Formation Proposal, to end with the full three-year CPS Implementation Proposal.

CLASSROOM PROPOSAL

In this section we describe our classroom proposal as a 45-90 minutes lesson, where students solve collaboratively a mathematics problem. We call this lesson *arpa,* as the Spanish acronym for problem solving activity in classroom (we use lower case italics, to distinguish it from ARPA). The goal of *arpa* is that all students solve a given problem and discuss their solutions in a plenary discussion.

Problem. A mathematics problem for *arpa* has to evoke the interest in students, it has to be a challenge, and students should feel they can solve it. For a given class, it is very difficult to find a problem that would meet these (dynamic) requirements because of the diversity of students. For this reason, a problem usually is a set consisting of: the original problem, a simplified version and an extended one. Other desirable characteristics of a problem are wideness of contents and richness of strategies and representations. An example of a problem is the following:



Two babies. John and Lucy are two babies just born in the hospital. The nurse was setting up the wall weight chart, when all the digits fell down and mixed them up. The nurse only remember that Lucy weighed more than John, that they weighted between 2 and 4 kilograms and the weights were expressed with three decimals ¿Can you help to reconstruct the weights of Lucy and John?

Possible simplifications: Give the students cards with the numbers and, if necessary, allow students to use only the cards 3,3,4,0. Possible extensions: Exactly, how many possibilities does the nurse have? Or What if the weights could be greather or equal to 2?

The enactment of *arpa* can be separated in four steps: delivery, activation, consolidation, and plenary discussion. Activation and consolidation have diffused boundaries, they overlap when students go back and forth. Next, we describe each of these steps.

Delivery. During this step, the teacher organizes students in random groups (Liljedahl, 2014), with four students per group. Next, with or without brief motivation words, the teacher offers the problem to students, preferable by means of one copy per student.

Activation. In activation, students read the problem and discuss its understanding, within their groups. Students may call the teacher who will give the group orientation through questions, not generic ones, but connected to the problem. At a point, students understand the problem and they start thinking about strategies to solve it. They work on the problem with their knowledge and skills, collaborating, making decisions, reasoning, etc. Occasionally, the students in a group get stuck or they need to ask a question. In either case, they call the teacher who will give them orientation through questions. When students get stuk, the teachers may take another action, that is to give students in the group a simplified version of the problem. Along activation (and consolidation), the teacher stimulates collaboration among group members and he/she interacts with the group as a whole, not with subgroups or individuals.

Consolidation. Once the members of a group feel they have solved the problem, they call the teacher. He/she will ask all students, in a dynamical way, to explain their solution and their procedures. If one student cannot explain its solution, the teacher will step aside, since a problem is solved only when all members can explain it. It may happen that students have made a mistake, then the teacher will ask guiding questions to help them to realize it, and then the group goes back to activation. Finally, it may happen that all members of the group explain the solution and procedure correctly, then the teacher gives them an extension of the problem and leads them to re-enter activation.

Plenary discussion. Once all groups have completed the solutions to the problem, or at least a simplified version of it, plenary discussion begins. Along this step, some students will take the blackboard and explain their solutions and procedures to their peers. All the time, the teachers will stimulate the discussion among the students with questions about the presentations, strategies, or concepts involved in the problem and solution (Stein et al., 2008). The teacher does not participate directly in the discussion.

The idealized description that I have given for *arpa* can be successfully run by a well-prepared teacher, equipped with all necessary competences, both in mathematics and pedagogy. Moreover, such a teacher is expected to be able to made a good problem choice, design its simplification and extension, to know the solution and various strategies that students may take, to know possible mistakes, to have thought about questions for students and, finally, she/he is expected to have enough flexibility to deal with situations that may emerge along the lesson. For having teachers to lead an *arpa*, it is not sufficient to let them follow a well-designed lesson script. From our experience, the formation of an in-service teacher to move from his/her position to an experienced *arpa* teacher is long, since it entails to change classroom practices and embrace a new pedagogical methodology (Swan, 2011; Zaks & Koichu, 2022).

CPS IMPLEMENTATION PROPOSAL

The implementation of CPS that I am describing is carried out along a 3-year PD program. In this section, I describe the first year and later I present the other two years.

The goal of the first year of PD (PD1) is to have teachers learn and experience classroom practices for running *arpa* with their students (in one class) and also let students become familiar with this way of participating in the lesson, changing teacher-student and student-student relationships. PD1 was originally designed by Perdomo-Díaz and Felmer (2017), inspired in the work of Koellner et al. (2007) and Nevado et al. (2015), taking in account the characteristics of an effective PD (Garet et al., 2001; Desimone et al., 2002; Marrongelle et al., 2013). Of note is that teachers that participate in our programs are not individual voluntary teachers, but they are enrolled by the institution, school or group of schools, with the purpose of implementing CPS. PD1 is an annual workshop, where teachers are organized in groups of 15-24 participants, meeting every month with a facilitator, in 3-hour sessions. At the beginning of the year, the sessions focus on PS activities but then the focus moves gradually to planning, implementing, and improving *arpa*. In addition, PD1 is based on the principles of *doing* and *reflecting* (Marrongelle et al., 2013), emphasizing its eminently practical approach.

During PD1 sessions, the facilitator stimulates teachers, promoting their teaching abilities and autonomy by using questions to answer questions, by promoting reflection, discussion, and the habit of justifying statements. The facilitator models teacher-student interactions through facilitator-teacher interactions (Prediger et al., 2019). Here the idea of *onion peels*, brought to us by Liljedahl, has been illuminating. Between two monthly sessions, teachers implement an *arpa* with their students. In this way a *cycle of improvement* (Koellner et al., 2007) is established: teachers plan *arpa* with facilitator support, then they implement *arpa* with their students and in the following session they reflect on their results, difficulties and achievements, and the cycle starts again, repeating it seven times along the year. In each cycle, aspects of the classroom practices are added, reflection among peers is developed, and the quality of *arpa* is improved. During PD1, the teachers keep a diary for recording their impressions on each implemented *arpa*, and the facilitator provides feedback to it. Along the year three *arpas* are video recorded, which are also fedback by the facilitator.

Throughout the first two sessions teachers work in various PS blocks in random groups and they participate in plenary discussions on emotions, strategies, and the role of facilitator. During the next four sessions, the facilitator and teachers model *arpa* (the facilitator acts as the teacher and teachers act as students) as a crucial learning opportunity. After the third session, teachers enact *arpa* in one of their classes, repeating it seven times during the school year. The sessions combine analysis of the experiences occurring in an earlier *arpa* and planning the next one. Teachers learn how to plan questions to ask their students, prior to *arpa*, in order to anticipate situations that may occur in classroom. This process may be reminiscent of the Lesson Study (Isoda, 2010).

Overall, the teachers are expected to learn how to conduct *arpa*, including working with groups, questioning, giving students time to work by themselves so they can make conjectures, mistakes and discuss with their peers. Once the basics of *arpa* have been mastered, teachers also learn how to adapt mathematical activities into challenging and reachable problems for their own students (Perdomo-Díaz & Felmer, 2017).

FORMATION OF FACILITATORS PROPOSAL

When a PD course or program is successfully implemented, its creators may wish to extend it to a larger number of teachers. However, it may be difficult for them to do it with their original team, so that new facilitators should come into play. The question is, how to attain new facilitators to have the success of the original team?

On the one hand, new facilitators need to understand the principles of the PD and they need to internalize and enact its procedures and proposed classroom practices. On the other hand, they have to be flexible to face teachers and contexts, other than those encountered by PD creator. Thus, new facilitators have to balance integrity and flexibility. But they have more challenges; since they are teachers they will go through an identity change process and they need to develop trust and respect from their peers (Valoyes-Chávez et al., 2022). New facilitators have to go through an extended time engagement with the principles and activities of the PD to enhance their knowledge and abilities to replicate it (Jackson et al. 2015). Our Formation of Facilitators Proposal takes approximately three years and is comprised of three stages that we present next.

Initial Formation for Facilitators. This initial stage set up the basis for initiating the role of a facilitator of a workshop. We have tried two forms of achieving this initial stage. The first one is having new facilitators to participate as teachers in the annual PD1. This experience provides them with knowledge about PD1 sessions and materials, and gives them practical knowledge for implementing *arpa* with students.

The second form is by having new facilitators to participate in a workshop led by an expert facilitator. This workshop has 3-hour sessions, twice a week, for six weeks. During the workshop, the participants get familiar with PD guidelines and resources. They have opportunities to experience *arpa* modelled by an expert facilitator and to model key facilitator's practices, such as asking questions, leading discussions, reflecting on their practices, and planning *arpa.* Research provides evidence that focusing on new classroom practices and materials is critical for enhancing facilitators' abilities and knowledge to conduct PD (Jackson et al., 2015). However, this workshop has an important drawback, the lack of actual practice with teachers or students. We mention here that the case studied by

Valoyes-Chávez et al. (2022) is of a teacher that followed this workshop and the two next stages of the formation proposal.

Guided Practice. The second stage of our formation proposal is a yearlong experience as a facilitator of PD1, with the close guidance of an expert facilitator. The new facilitators meet the expert facilitator monthly, in sessions where they reflect on their last PD1 session and they plan the new PD1 session, in a *spiral of improvement*. During the year, the expert facilitator models some practices, the use of tools and resources to help the new facilitators to move towards their autonomy. These practice-based and job-embedded activities, guided by an expert facilitator in a reflective way, seem to contribute to fostering facilitators learning (Roesken-Winter et al., 2015).

Autonomous Practice. This last stage in the formation proposal is devoted to consolidating the facilitators' roles and practices. During this stage, new facilitators will again lead PD1 or a similar workshop along a year, but obtain more autonomy. They meet monthly for planning the workshop sessions, but only in three of these sessions they are accompanied by an expert facilitator. Expert facilitators will also answer questions that facilitators may have by email or phone.

We believe that an approach to facilitator formation as presented here may provide new facilitators with the essentials for being successful in a process of extending a PD course or program, even though our experience is limited to two cohorts of facilitators.

CPS IMPLEMENTATION PROPOSAL (CONTINUED)

The CPS Implementation Proposal is a 3-year program having as its main goal the systematic enactment of CPS in school, at all levels considered in the plan. This means that *arpa* is implemented regularly at all the levels considered, with a given periodicity, and with material (problems) provided by the program. In order to achieve the main goal of the program, PD1 is conducted in accordance with the description given in the earlier section, followed by PD2 and PD3 that I discuss below.

At this point, it is important to note that it would not be realistic, and not even desirable, that a program is implemented without changes, that is, keeping all prescribed steps and planned times. Three years is a long time, in which teachers, principals and school culture will interact with the program. Moreover, emerging local events may enhance or hinder the progress of the program. Just as examples, in Chile we occasionally encounter long teacher strikes that require reorganization of PD activities, or the change of the school principal that could low down or give an impulse to the program. Moreover, for an innovation to be implemented in a school, it has to be adapted to school realities in order for the program to be sustainable. We recall here the definition of sustainability given by Prenger et al. (2022) in their review paper: "Sustainability refers to the process of integrating the intervention's core aspects in organizational routines, which are adaptive to ongoing work" (p. 14). This definition provides a dynamic interpretation of sustainability, taking in account that the innovation has to be adapted so its core practices are integrated into existing ones.

PD2 Proposal. The second year of the program has three main goals. The first one is to improve and introduce new *arpa* practices and to consolidate the use of *arpa* once a month. The second goal is to extend *arpa* to all students in the classes that each teacher teaches, not only to the students in the designated class considered in PD1. To achieve the first and second goal, teachers meet monthly with a facilitator in 3-hour sessions implementing the *cycle of improvement*, similar to PD1.

The third goal is to construct a Problem Book composed of mathematics problems for each school level involved in the program. The purpose of the book is to prepare the way for the incorporation of *arpas* along the year in the Annual Lesson Plan (ALP), so to institutionalize the innovation. The construction of this book requires the concourse of facilitators, teachers and school pedagogical leaders (depending on the organization or size of the school, they may include the head department, the teaching coordinator or even the principal). Their first decision concerns the frequency in which *arpas* are to be regularly implemented during the third year and thereafter, which determines the number of problems needed to be constructed (or adapted) for each level. As indicated before, the problems should be interesting for students, and correspond to challenges that the students would feel they can solve. Moreover, the problems should be aligned with the curriculum, context, and school culture. Even though, the book construction should involve teachers and school pedagogical leaders, in our experience this process is so complex and time consuming that the main part of the work relies on facilitators. With the Problem Book at hand, the ALP is designed, identifying which lessons will be replaced by *arpas* and which problems will be used in them.

PD3 Proposal. The third year of the program is devoted to the first implementation of the ALP with all teachers and students involved in the program. During PD3, monthly sessions for teachers are held, where the facilitators accompany teachers in their planning of *arpa*. In the best case, when teachers have participated in the problem adaptation process, they know the problems, but even then, the teachers would be greatly benefited from the facilitator support when planning *arpas*. The goal of PD3 is to have teachers implement the ALP with *arpa* as a regular activity and to help them to create a community of practice able to sustain the CPS.

THREE CASES OF IMPLEMENTATION OF CPS

This section is devoted to three cases of implementation of PD programs, following the CPS Implementation Proposal given above. But before that, we give ideas about PD funding in Chile and some quantitative parameters of the ARPA Initiative.

Professional Development in Chile. Most PD in Chile is funded by the state through schools, assigned according to the number and socioeconomical index of their students. Within this funding system, a university program like ARPA Initiative has to reach schools leaders and offer them a PD. Once these leaders accept the offer, an agreement is signed, however this agreement has to be signed year by year. This funding system has various disadvantages for programs like the ARPA Initiative, however, it has at least two advantages from the school principals and teachers' perspective. First, the programs have to fit quite well to school's needs and second, the annual funding allows the program recipients to evaluate the advances every year, and terminate the program if it does not reach the expected results. Within this funding system, a program like ARPA Initiative gets funds for development, but research funding has to come from other sources. Thus, most of the research performed regarding the implementation of ARPA, has been fragmented and directed only to certain aspects of this initiative.

Some ARPA numbers. Along the almost ten years from the creation of ARPA Initiative, many workshops for mathematics teachers have been implemented, reaching about 10,000 teachers teaching at different school levels. Among them, about 6,000 teachers participated in short workshops (3-4 hours), 1,600 participated in summer workshops (25 hours), 2,000 participated in annual workshops, 250 in two or more years workshops and 110 participated in initial formation for facilitators workshops.

Now, we present three cases of PD programs aiming to incorporate CPS in classrooms in different contexts, in order to illustrate the flexibility and adaptability that the program needs to have to get to desired results.

Ránquil County. Ránquil is located in central Chile, 500 km south of the capital, with a population of almost 6,000 inhabitants. Its educational system is managed by the county government and consists of 9 schools, 7 of them in rural areas.

Fundación Educacional Arauco (FEA), in collaboration with the ARPA Initiative, conducted the Mathematics Improvement Program between 2017 and 2021 (Fundación Educacional Arauco, 2022). During the first two years, the ARPA Initiative provided PD1 and PD2 to almost 60 pre-school, elementary, and middle school teachers. Along the program, facilitators from FEA run a formation plan for school leaders and they visited classrooms providing teachers support. In October 18th, 2019, a national social outbreak took place (Freire, 2020), followed by the Covid Pandemic in March 2020. These events disrupted the program which was resumed in 2021, but only partially.

For evaluating the program, pre- and post-questionnaires for students and teachers and video-recorded lessons were considered. Six items, using a Likert scale with 1 to 5 points (from fully disagree to fully agree), were used to search for students' self-efficacy and enjoyment-interest for mathematics. With a sample of 333 students, it was found that students having a 4 or 5 in self-efficacy items increased by 5% and those having a 1 or 2 decreased by 1%. These modestly positive results are complemented with better results on enjoyment-interest in mathematics:

students having a 4 or 5 in these items increased by 17% and those having a 1 or 2 decreased by 11%.
The participating teachers were given pre- and post-questionnaires on their perceptions on promotion of students' skills. With a sample of 36 teachers, data showed that teachers perceiving that they promote always or almost always communication and argumentation skills increased by 35%, representation skills by 37% and PS skills by 27%. In addition, 16 teachers where video-recorded in one lecture at the beginning and end of the program. The videos were rated in three dimensions with four levels of achievement and it was found that teachers raised their performance in encouraging: communication and argumentation skills, representation skills and PS skills.

The positive result on students' perception of self-efficacy and interest-enjoyment of mathematics may be attributed to the implementation of *arpa* in mathematics lessons. This is consistent with teachers' own perception as promoters of communication, representation, and PS skills, which is also consistent with actual performance of teachers observed in video recorded lessons (Fundación Educacional Arauco, 2022).

Rengo County. Rengo is located 100 km south of the capital, with 60 thousand inhabitants. In 2017, a PD program funded by the county government was initiated in 22 schools.

The program began with an Initial Formation of Facilitators workshop in April 2017, which was attended by 25 teachers. Among them, three new facilitators were recruited and continued their professional development along the program. The actual PD program, with the participation of about 60 elementary and middle school teachers, was operated between August 2017 and July 2019, with PD1 and PD2. During the second half of 2018, the construction of the Problem Book and the ALP took place with the labor of an ARPA team who consulted with school leaders and some of the teachers. In August 2019, the implementation of the ALP was initiated, but the program was disrupted by the national social outbreak in October 2019. Then the pandemic emergency definitely discontinued the program in March 2020.

Since the program ended before it was completed, a final evaluation was not possible. Through internal team reflections three main difficulties were identified: coordination between the program and schools, low commitment of some of the school leaders and resistance from many of their teachers. Nevertheless, the program was appreciated by some teachers, who incorporated *arpa* as a regular activity. The difficulties in the implementation were mainly associated to the lack of experience of the ARPA team, specially in managing the expectations and commitments at the begining of the program and in coordinationg a sizable number of schools.

Even though a formal evaluation of the whole program has not been made, an assessment with a limited number of teachers and students was carried out by

Saadati and Felmer (2021). Among the teachers participating in the program, five 5th grade teachers were invited to participate in the study, with their 100 students. Two classes with 45 students made up the control group with traditional teaching and the other classes with 55 students made up the experimental group, with 6 or 7 *arpas* along the year, following PD1 guidelines. Out of this sample of 100 students, 47 were girls and 53 were boys. Data were collected in October 2017 and October 2018. Each student of the sample participated in a pre- and post-tests and obtained a performance score for each test. An independent sample *t*-test showed that there was no significant difference between the control and experimental group in the pre-test. For the data collected in the post-test, a covariance analysis was conducted to compare the performance scores of the groups. After adjusting for the pre-test scores, we found a significant difference between the control and experimental group on the performance score (see Saadati & Felmer, 2021, for details). This study may be an indication that mathematics teachers participating in the CPS workshop can change their practices implementing *arpas* and improving PS skills of their students.

INACAP. Among institutions providing *technical post-secondary education* in Chile, INACAP is the largest one, with 27 campuses along the country. Every year, around 40,000 students enter INACAP and during the first year, all these students have a mathematics course. In what follows, we describe the process that led INACAP to implement CPS in all first-year mathematics courses. Our description divides the process in five periods.

Period 1. The process started in 2016 with the implementation of PD1 in 2 campuses, followed by the implementation of PD1 in 4 campuses in 2017. At this point, the results were promising, since participating teachers were able to implement *arpa* with their students however, the subsequent formation path was not clear for them.

Period 2. After a reflection with the institution, two goals were set for 2018: design problems for the implementation of *arpa* and formation of facilitators from INACAP. Thus, a problem co-construction process was carried out and teachers from 2016 cohort implemented *arpa* along the year, with support of a facilitator. In practice, PD2+PD3 was performed during 2018. In addition, a 36-hour facilitators workshop was designed and carried out during 2018, with the participation of 12 teachers, two from each of the six campuses that had PD1 so far. The goal of the workshop was preparing campus leaders to keep and improve the implementation of *arpa* in their contexts.

Period 3. During 2019, a scaling plan was designed for implementing PD1 in all remaining campuses during 2020 and 2021. This plan came together with the formation of an ARPA-Inacap Team (AIT) inside the institutions with the aim of leading PD implementation and material construction. The members of AIT were selected among teachers of best performance in PD1 (2016-2017). This team continued their formation process, consisting of a Guided and Autonomous Practice for the next two years.

Period 4. The national social outbreak and the hit of COVID pandemic in March 2020 postponed the 2020-21 scaling plan since face-to-face educational activities were prohibited. Then, an adaptation process was set up for designing an e-learning version of PD1 using meeting platforms for PD1 sessions and teachers' implementation of *arpa*. Thus, the scaling plan with PD1, was carried out between August 2020 and July 2021 for 10 campuses and between March and December 2021 with teachers from the remaining campuses was resumed. During 2022, another cohort of teachers followed PD1. In three year, more than 240 mathematics teachers from INACAP received PD1 lead by AIT facilitators who also completed their formation as facilitators.

Period 5. Having all mathematics teachers from INACP completed PD1, one would expect they continue with PD2 and PD3, as CPS Implementation Proposal suggests. However, the institution required the ARPA Initiative to design and implement a Certification Program for teachers having completed PD1. This program defined a quality standard that teachers should achieve for certification: a teacher must exhibit knowledge of *arpa* concepts, implementation of *arpa* of quality, and evidence that their students identify *arpa* practices. It is interesting that this certification program can provide evidence of sustainability of the program, that we hope to report in the future.

As a synthesis of the seven years of experience with INACAP, we describe a latest material design process carried out by AIT. With the goal of providing teachers with a material with coherent teaching practices, AIT designed a complete lesson plan to be installed during 2023. The new material proposes three different active learning activities: *arpa*, collaborative learning activities (Laal et al., 2012), and out-of-class independent work. In collaborative learning activities, students are organized in random groups and teachers act as in *arpa*. However, in these activities, students receive guided materials with instructions, activities, and questions, allowing them to learn new concepts, relations among concepts, and mathematics principles. This collaborative learning activity differs from *arpa*, since it does not provide challenges to students (problems), but collaboration and autonomy are explicitely encouraged.

The evolution of CPS implementation program due to institutional requirements and development of AIT designing material as an adaptation of *arpa*-problems to a consistent lessons plan is an example of transfer of knowledge and know-how and of the institutional adaptation for sustainability (Prenger et al., 2022).

CONCLUDING REMARKS

The proposals for CPS implementation that I have presented was informed by research and constructed from experience. In this respect, the extensive collaboration with INACAP, described above, has become the most important CPS implementation process for us in Chile and also a fundamental source of learning. Our proposals include the participation of school leaders, but we prefer not to elaborate on this topic because of the limited experience we have on it. However, in our experience, we have seen an increasing need of incorporating them in parallel workshops, where they get prepared to support and provide feedback to the teachers. This is crucial for the sustainability of the innovation, since school leaders may keep integrity of *arpas* and introduce new teachers into the practice of *arpa*.

The conceptualization of PD with the three-tetrahedron model (Roesken-Winter et al., 2021) and by means of the *implementation chain* (Koichu et al., 2022) appear as a valuable opportunity for analysing different aspects of our proposals. These are certainly interesting issues for the newly created research journal IRME.

The COVID pandemic had a strong impact on ARPA Initiative, not only because our classroom and PD proposals are best implemented face to face, but since teachers and principals were unavailable for PD at that time. The post-pandemic period is opening new horizons for CPS implementation. Our proposals and our entire approach to PD is marked by *flexibility*. In defining a PD program with institutions and during the implementation process, flexibility is required to account for teachers and school's needs and to confront the unexpected events. Moreover, flexibility is crucial for allowing the adaptation of the innovation, a key element for sustainability. We conclude: *Things never turn out as planned, and this is part of the game.*

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MATHEMATICAL SUBJECTIVATION: DEATH SENTENCE OR CHANCES FOR A TERRESTRIAL LIFE?

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Moving away from the desire to solve current problems with/through mathematics education, the question of how mathematics education is entangled with "climate change" invites into an investigation of the promises and limits of our field of practice and research. Mathematics as a network of forms of knowing and acting on the world is at the core of the project of Modernity. Mathematics education has been increasingly central in governing towards desired forms of subjectivity that sustain and tries to achieve the promises of Modernity. The project that has allowed (Western) humans to tame, control and exploit nature is the very same project that has worn out human and non-human existence to the verge of its destruction. A mathematics education that desires and continues this project risks proclaiming a death sentence to all forms of existence in the planet. Can mathematics education be reconfigured in other directions to support the chances for a Terrestrial life? On which coordinates could such reconfiguration build? What could be its promises and pitfalls?

RESISTING THE EDUCATIONALIZATION OF CLIMATE CHANGE

Historians of education have proposed that one of the key characteristics of education in Modernity is what they call the "educationalization of social problems" (Depaepe & Smeyers, 2008). The invention of the child as a subject of education —a pupil or a student— goes hand in hand with the institutionalization of education and with the strong belief that the intervention on the child through regular, planned and intentional instruction is a desirable and effective way of addressing problems in society. In simple words this means that it is characteristic of Modern societies to explicitly use education to mould children —their bodies, mind and behaviour— towards correcting current identified social ills or preventing future possible problems. A simple manifestation of this characteristic is to be found in the well-known tendency to create new school subjects to address perceived lack of knowledge in the population associated with particular crises. Another manifestation is the steering of curricula —at any level of education— towards the provision of qualifications for anticipated future needs of individuals, society or the labour market. This is evident in calls such as education for fostering 21st century skills, to secure the adequately equipped human capital suitable for the workforce in the knowledge economy (e.g., van Laar et al., 2017).

Mathematics education has not been immune to this logic. Indeed, mathematics education as a State regulated area of the school curriculum in Modern times has

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 53-68). PME 46.

been perceived as a key player in addressing different kinds of social problems: Arithmetic skills should be provided to train the necessary workforce to advance a scientific industrialization and specialized state bureaucracy (e.g., Kollosche, 2014); New Mathematics to strengthen the West scientific and technological capacity at the core of modernization and democratization to defend and promote the progress of Western culture (e.g., De Bock & Vanpaemel, 2019); or even more recently renewed on-line mathematical instruction to address the increase in socio-economic and health inequalities evidenced by the COVID-19 pandemic lockdown of schools and societies (Ziols & Kirchgasler, 2021). An interesting overview of the problems and areas that mathematics educators think should be addressed in the decades to come is presented in Bakker et al. (2021).

Following this culturally learned way of thinking about education, it is not surprising that the eminent environmental crisis —that we can by now set under the umbrella term “climate change”— also triggers our desire of positing mathematics education as a possible contributor to a remedy. In this case, we would not only be following the impulse of the educationalization of social problems, but what Tröhler (2017) has called the educationalization of the world. Local, regional and international conferences of mathematics education (research) have called participants to reflect on “una educación matemática para el cambio [a mathematics education for change]” (e.g., National conference of mathematics education in Chile, JNEM XXIV), and PME 46 congregates us around the topic of “mathematics education for global sustainability”:

> This theme calls attention to caretaking of human wellbeing, coping with global disease, narrowing economic and social inequalities, and mitigating climate emergencies. These crises can be seen as interwoven threads that are inseparable one from another.
>
> Mathematics education holds particular promise for crisis resolution towards sustainability. (https://pme46.edu.haifa.ac.il/)

As committed (mathematics) educators, we are ready to take the task of solving current social and world problems by improving what we best do: the teaching and learning of mathematics. Since, if mathematics is important as an area of the school curriculum, it should keep on providing current and future qualifications for individuals, communities and nations to advance towards a better future (e.g., Bakker et al., 2021). Mathematics education has to keep on responding to changing social needs and fulfilling its promise of… of what? Hold on a moment…

The notion of educationalization has offered historians of education a critical lens to examine and problematize the political in education by contextualizing educational desires that govern policies, curricular reforms and practices in the larger network of political, economic and cultural transformations within which education always functions. This move has offered understanding of the multiple double binds of education and its contradictions. On the one hand, education is conceived as being in constant change to respond to current problems and to face future problems; always offering a hope for a solution towards a better and improved situation. But on the other hand, education also stabilizes, reproduces and even legitimizes existing patterns of behaviour and relations in collectives. Labaree (2008, p. 448) argues that recognizing how educationalization works allows to see how:

> We assign formal responsibility to education for solving our most pressing social problems in light of our highest social ideals, with the tacit understanding that by educationalizing these problem-solving efforts we are seeking a solution that is more formal than substantive. We are saying that we are willing to accept what education can produce —new programs, new curricula, new institutions, new degrees, new educational opportunities— in place of solutions that might make real changes in the ways in which we distribute social power, wealth, and honor.

In other words, while productive efforts in education may mobilize some changes, significant changes to the distribution of power and resources in collectives require more than education to be achieved. Thus, thinking with Labaree, we could suspend for a moment our impulse and "highest ideals" of contributing to solving the multiple crises of sustainability now, since it is quite likely that embracing new pedagogical methods, different didactic approaches and more research to accompany them may lead to "more formal than substantive" change. Instead, let us consider the ways in which the institutions and networks of mathematics education may be entangled in the distribution of "power, wealth, honor" and survival in the current configurations of "climate change". In my view, this consideration forces us to problematize mathematics education as a field of and research to critically examine its entanglement with the pressing situation of a world that gets closer the verge of crumbling down.

Rather than proposing mathematics education of a certain type to solve the problem of sustainability, we could ask the question of how mathematics education is implicated in the current situation itself. That is, is there a chance that mathematics education contributes to the problem and not just to a solution? The exploration brings us outside of the comfortable core of our usual concern with understanding, designing and improving teaching and learning, and casts us in the cultural politics of mathematics education (e.g., Diaz, 2017; Valero, 2018). For this, we need to displace our investigation into cultural studies and philosophy in order to come up with tools and ways of thinking that may offer other ethico-political insights and coordinates for action.

I will argue that one of the key functions of mathematics education in Modern, state regulated times is promoting mathematical subjectivation. Mathematical subjectivation can be seen as a powerful form of governing of individuals and communities to manage, advance and sustain modernization in different spheres of life. This form of understanding the self and the human in culture is at the centre of the "climate change" crisis. "Climate change" —in the Latourian sense of the irreversible transformations in the "relations between humans and the material conditions of their existence" (Latour, 2018, p. I)— evidences that the project of

Modernity and its modernizing impulse has overpassed its limits. Thus, maintaining and promoting Modern forms of mathematical subjectivation risks proclaiming a death sentence on the planet, since the planet cannot bear any further project of modernization. The ethico-political responsibility of mathematics education means also exploring different reconfigurations of "the mathematical" that may lead to other subjectivities. These may provide a chance for a Terrestrial life for humans and non-humans alike.

THE POWER EFFECTS OF MODERN MATHEMATICS EDUCATION

The educationalization of social problems, as a notion and as a phenomenon, highlights the power of education: Education *does* something to the people involved. Depending on the theoretical perspective from which we choose to approach it, we would phrase this *doing* in slightly different ways. From a cultural political stance inspired in Michel Foucault (1972; 2007), it can be argued that an important invention of Modernity was the controlled and regulated use of education as an effective way of creating changes in skills, knowledge, behaviour and moralities in people, and for planting new sensibilities and habits of mind and spirit, to build forms of economic and political organization in society (e.g., Tröhler, 2011). Education is entangled with political power as part of the many technologies that in Modern, knowledge and expert based societies, have been deployed to govern the population (Ball, 2017). In this sense, education organized and administered through school as an institution makes *types of people*, tracing directions for what and how populations should know, behave, and become (Popkewitz, 2008). Education is therefore central in subjectivation. The notion of subjectivation (e.g., Rosenberg & Milchman, 2009) directs attention to how individuals, in historically formed arrangements of culture and power and in constitutive connection with other people and with things, create ways of being, understanding and performing themselves. That is, the sense of who we are and could become emerges inside a collective, in the present with its inherited past and potential possible futures.

Mathematics education, in its current form, is sustained as a core area of the school curriculum. As such, it is inseparable from the historical and institutional forming of Modern schooling, and the whole of its technologies for the governing of populations and the subjectivation of individuals. In the last three decades, researchers have studied mathematics teaching and learning in their broad societal/cultural/political, institutional contexts (e.g., Kollosche, 2016; Planas & Valero, 2016). Common to these studies is the recognition that mathematics education does something with/in "learners", that goes beyond transformations in their cognition and thinking around mathematical notions and procedures. Mathematics "objectification" also has an effect of subjectivation. This is, for example, a central tenet in Radford's (2021) theory of objectification. Being the other side of the coin of the objectification of the culturally shared forms of mathematical knowledge, Radford sees subjectification as the accompanying constant process of becoming: “through mathematical ideas and in dealing with others, students and their teachers co-produce themselves and, at the same time, are produced by their cultural-historical context” (p. 185).

Another related area which converges in exploring mathematical subjectivation are studies on the processes of mathematical identity formation and identification (e.g., Darragh, 2016; Graven & Heyd-Metzuyanim, 2019). From the variety of theoretical stances adopted to conceptualize identity/identification, of sites and practices being studied, and of methodologies deployed, the research shows that “the key to our identities lies in our inclusions in, and exclusions from, different communities, and in our performances within those of which we are members.” (Sfard, 2019, p. 560) In situations of learning and teaching mathematics, multiple dimensions of being, thinking and knowing are entangled in institutional discourses. Learners come to grips with mathematical notions, while they form a sense of themselves in relation to categories of racial, class and gender differentiation that are activated in mathematics classrooms (e.g., Gholson & Martin, 2019). Identity research documents that mathematics education leaves marks in children and learners, that are more complicated than constructing culturally shared forms of mathematical knowledge and desired mathematical competences, skills and qualifications of use for future studies and the labour market (e.g., Valoyes-Chávez & Darragh, 2022).

We know that mathematics education subjectifies, or at least, that research has provided both theoretical arguments and empirical evidence that this is the case. I have suggested that the subjectivation can be conceptualized as a power effect of governing school organized mathematics education. The issue remaining is which kind of subjectivities mathematics education mobilizes. In other terms: What type of subject do we become with and through the school organized practices of mathematics teaching and learning? Which values and traits are inscribed in learners as they participate —successfully or not— in the daily practices of mathematics classrooms for the many years of compulsory school? Put it in very simple terms, after many years of sitting in a mathematics classroom and when for most people the details of the concepts and procedures stipulated in the curriculum have been forgotten, what does remain in us as a result of the workings of carefully planned and steered mathematical pedagogies?

TRACING MATHEMATICAL SUBJECTIVATION

The mathematically abled subject

The systematic and sustained daily routines of mathematics education generate “principles about what is seen, talked about, felt, and acted on” that fabricate mathematical abled bodies (Yolcu & Popkewitz, 2019). A mathematically abled subject distinguishes different objects and classifies them in categories, as well as orders and ranks them according to some criteria. The subject identifies patterns across different cases and express them as generalities. The subject thinks in rational ways, adopting the rules of step-by-step arguments that lead to secure and certain statements. The subject detaches emotionally from these objects to create a safe distance that allows to operate on them to simplify them, control them and act with/on them. Once enough distance is achieved, the subject moves from a concreteness and situatedness of some objects towards more abstract and general objects and operations. Not only does the subject treat these objects as neutral, but also perceives their own action as either neutral or necessary. This is, once the operation of/with mathematical objects takes place, there seems to be some kind of inevitable, natural chain of operations that unfold. The subject desires to identify problems and engage in action that solves the problems; and those solutions are perceived as good and desirable —as desirable as the very same problem-solving impulse that initiated them. The subject trusts in the authority of the reasoning carried out with and of results produced from operating with the elements of mathematics —numbers, symbols, graphs, etc. But at the same time, the subject desires to improve and find better and smarter solutions, through self-regulation and evaluation of the own performance. The subject appreciates all of the above and ascribes differential value to the bodies —often referred to in terms of "minds"— according to a ranking of displayed ability. The subject recognizes the economic value that these capabilities can be traded for.

Rationalism, objectivism, abstraction, neutrality, deduction, agency and entrepreneurship, improvement, progress, growth, optimization, meta-reflection, trust in the authority of mathematics, appreciation and positive attitude to mathematics, differentiation of (economic) value, comparison, and competitiveness are just few of the characteristics of mathematical subjectivation that one could identify from reading broadly in the expanding research adopting critical and political perspectives in mathematics education (e.g., Jurdak & Vithal, 2018; Jurdak et al., 2016). Some of these characteristics have been called the values and affects of mathematics education (e.g., Seah et al., 2016), and are often associated with the de-humanizing of mathematics (e.g., Peck, 2018) and the need for ethical and spiritual turns in the field (e.g., Boylan, 2016; Gutiérrez, 2022). These characteristics can also be traced in the bulk of research in mathematics education. Knijnik and I have argued elsewhere (Valero & Knijnik, 2015) that all forms of mathematics education conduct students and teachers in a given direction, even the types of education —and research— that openly declare being a-political. As mathematics educators, we desire to contribute fabricating mathematical subjects who understand themselves and the world with and through mathematics. Thus, these characteristics are quite familiar to our community since they constitute the "epistemic virtues" —Daston and Gallison's (2007) notion to think about the subjectivation effect that accompanies scientific objectivity— that we strive to achieve.

These epistemic virtues, values or characteristics of mathematical subjectivation connect individuals with the culturally produced forms of reason, doing and being of Modernity. Modernity as a cultural aspiration for the human, installs the belief on science-based reason with a universal, emancipatory capacity for changing both the natural and social/cultural world (e.g., Gaukroger, 2006). Human agency, the hope for scientific-guided progress, and the desire of planning the future both creates and requires a type of subjectivity that places “individuals in a relation to transcendental categories that seem to have no particular historical location or author to establish a home” (Popkewitz, 2008, p. 30), that is, a *homeless mind* (Popkewitz, 2008, p. 29).

Inscription devices and mathematical subjectivation

Here there emerge two questions: How does mathematics education produce the sense of a homeless mind? And why is this a key point in the continuation of the cultural project of Modernity? An example of school geometry comes very handy here. Despite controversies of the extent and form that Euclidean geometry can and should occupy in school curricula, it is a persistent area that has a pivotal functions for children to “experience the making and proving of conjectures […] develop logical reasoning […] learn a language that allow them to model the world [… and] acquire tools for future non-mathematical work” (Sinclair & Bruce, 2015, p. 320). Based on the analysis of the Chilean school curriculum and its proficiency tests system, Andrade-Molina and Valero (2017) examine how geometry teaching is designed to unfold in leading learners to achieve the aims of the curriculum for geometry. The training that classroom pedagogy sets in motion, which is also assessed in proficiency tests that correspond to curricular established levels of competence, follows a carefully designed, ideal map of progress. Starting by “acquiring the basic elements of geometrical figures and prisms and realising the usefulness of the squared paper, students should learn how to measure, depict, and operate with these tools.” (p. 256) Passing through each level, new acquired concepts should become useful to apply in the subsequent level that progressively increases in complexity of the concepts, tools and expectations of connection with real life situations. At the end of the last level, the seventh, students should be able to navigate in space using “models as Cartesian and parametric equations” (p. 259). This type of training has two main effects:

> First, the eye of the body becomes a trained eye through the norms of Euclidean reason. […] Second, for this eye to “see”, a new type of space is needed. These technologies simultaneously fabricate a subject and objectify space in particular ways. The space of school geometry emerges. This space has been “chopped” into particular routines, and it has restricted ways of seeing and being. (p. 261)

The double effect on the subject who operates with figures, drawings, transformations and squared paper and on the very same idea of space in a Euclidean fashion —timeless, universal and decontextualized— is one instance and specific inscription of the “homeless mind” (see e.g., Valero et al., 2012).

From a different perspective focusing on the inscription devices of geometrical practice that have been deployed in the scientific enterprise, Latour (1986) has examined the multiple inscriptions performed with instruments, diagrams, mathematical models, pencil and paper as people discuss and argue to produce a scientific result. The latter emerges from the repeated routines and connections among scientists who have strived to act on the inscriptions and their combinations with the "epistemic virtues" that they have trained themselves to display. Inscriptions in scientific practice, Latour argues, allow mobility and translation from the concreteness and particularity of one site of practice to other situations and spaces. Once detached, the qualities of concrete material configurations are made ready for calculation and manipulation. This is the power of Modern scientific explanations: their detachment from the concreteness of the situated material configurations being explained (Latour, 1988), which generates the sense that the scientific outlook is "'a point of view from nowhere' that allowed disembodied and interchangeable minds to write laws applicable to the entire cosmos" (Latour, 2017, p. 77).

The notion of inscription device that I used above deserves further explanation. In the latter sense and drawing on Latour (1986), artefacts and tools mobilized in practices where knowledge is generated and explanations are articulated are used to leave registers and traces of the activity taking place. The inscriptions however become central to knowing as they can be "mobile but also immutable, presentable, readable and combinable with one another" (p. 7). Bringing them as combinable cascades of detached representations —that displace actions from particular objects to drawn or diagrammed new flat objects existing on a paper, drawing or computer screen— allow for calculation and manipulation in ways that generate a new form of power: the movable scientific explanation and its array of articulated inscription devices. The focus for Latour, however, is not just on the inscription devices but also on their functioning within material practices and in/with people.

It is here where the notion of inscription device relates to the former sense previously mentioned: that of the training that it takes for subjects to both generate and operate with inscription devices. Latour (1986) himself and also Popkewitz (2004) —drawing on Foucault and Latour— point to school as a space for such training. Schools are places where children learn to manipulate and organize the inscriptions of transmogrified academic knowledge —to use Popkewitz's term for the weird forms of "transposed" knowledge that school curricula make. Above all, children learn to "believe the last one [of the arrayed inscriptions] on the series more than any evidence to the contrary" (Latour, 1986, p. 26). In the routines of schools and the carefully orchestrated pedagogies of different mathematical areas and their concepts, and procedures, there emerge purposeful ways of working with children for achieving this purpose (see e.g., Diaz, 2017 for the case of the equal sign in primary school mathematics). It is here where the notion of inscription devices also covers the training that leaves marks on children's mind, body and being to think according to the particular patterns and logic of the Western scientific culture.

And here we come back to mathematical subjectivation. It is clear that the epistemic virtues and values of the Modern scientific outlook and of the mathematical subjectivation in the practices of school mathematics education leave traces on each other. In other words, the long-term moves in the articulation of a scientific outlook, in which mathematics is historically entangled, are sustained by a network of institutions, one of which is organized education and its state regulated mathematics curricula; these two co-constitute each other. The meticulous training of school geometry is one example of how in school there is an articulation of the inscription devices of geometry and the inscriptions devices of pedagogy. Operating together, in students' work in/for learning, they make possible a sense of self as a homeless mind to emerge. And even if Latour provocatively argued that "we have never been Modern" (Latour, 1993), we culturally privilege and strive to govern subjectivities towards the values and epistemic virtues of Modernity. It does not take much to provide evidence that this is the case: Suffice to notice the increasing alignment of national educational policies and mathematics school curricula with the framework of the Organization for Economic Co-operation and Development (OECD)'s Programme for International Student Assessment (PISA) to identify the direction of the governing of education in current times, and their effect on the network of practices and institutions of mathematics education (e.g., Giberti & Maffia, 2020; Kanes et al., 2014).

Of course, identifying mathematical subjectivation and its orienting values as I have attempted here is not equivalent to assert that there is a homogenous effective subjection of each single individual to turn into a kind of identically-factory-produced tin soldier. Indeed, our current post-factual times may raise doubts on whether this project has at all succeeded in grasping peoples' minds and bodies (e.g., Parra, 2021). Power and its effects of subjectivation always have cracks for freedom. And yet, following Popkewitz (2004), the notion of fabrication directs our attention to the "fabric" of practice and power which orient our becoming. This fabric is often invisible to us. I would still argue that the "fabric" —or network— in which mathematics subjectivation is entangled is strong and needs careful consideration in our current time of "climate change".

THE IMPOSSIBILITY OF (MORE) MODERNIZATION

To understand the current configuration of climate change only in terms of the catastrophic human destruction of the environment would be short-sighted. Latour (2018) turns the gaze to the current articulation of apparently unrelated events such as the revival and strengthening of populist nationalisms around the world, the deregulations of markets in globalization, the increase in poverty and inequalities in populations, the increase in migrations around the world, and the denial of climate change. Climate change is all of these events together, and they bluntly make visible that the relations between humans and "material conditions of their existence" have transformed (p. I). In this situation, new political orientations need to be constructed; however, on different coordinates since it seems that the desire for a common life and future to orient politics is long ago gone. Divisions and tensions are evident all around the globe, and the promises of a better life for all seem to be at stake. Thus, Latour invites us to understand the configuration of what he calls the "New Climatic Regime" (Latour, 2017).

The expression "New Climatic Regime" playfully indicates that the "New Regime" is in urgent need of revision. The New Regime was installed by the French Revolution and taken as a symbol of the cultural and political project the Enlightenment and Modernity. It replaced the "Old Regime" of political tyranny based on the authority granted by God to the rulers. In the New Regime, knowledge of scientific type, for men to take agency of their lives and to tame nature, was expected to bring a new form of government, a new order, a new hope for the future. This cultural project, expanded through will (or force) around the world, while creating the many improvements for (some) humans that we know as progress, has also left irreversible damage on all beings, human and non-human. The formulation of the New Climatic Regime points to the situation "in which the physical framework that the Moderns had taken for granted, the ground on which their history had always been played out, has become unstable" (p. 3). It recognizes that Nature is breaking down and about to collapse in a way that seriously threatens the viability of our forms of life. Most importantly, it draws attention to the fact that the ways in which the Western scientific culture and its expansion around the globe conceived of nature, science, politics and even the human are no longer sustainable. For example, Nature has been conceived as detached from culture — or the life of human. Nature was made the terrain for the generation of material conditions of existence. Nature was in the land and all its non-human elements that could be objectified as countable, measurable and manageable. They all should be appropriated and infinitely exploited. Nature was in the planet that scientific knowledge and technology should explain, model and thus help conquer. Science, mathematics and technology became the best tools for humans to reign over all species and Nature, to solve problems and to build prosperity through the unstoppable chain of growth —in all aspects of culture.

That the planet is turning against us is an urgent call to revise the Modern values, epistemic virtues and forms of existence that we so dearly hold to. We are daily confronted with the limits of the project of Modernity itself: As Latour (2017, 2018) argues, the forms of production, knowledge, exchange and life are called to question to the extent that no possible project of "modernization" of any nation, developed or developing, can be achieved as planned because there will be no physical Earth on which it can be unfolded. Thus, the ideas that have dominated the Western rationality and its expansion to drive humans towards progress and to reach more desirable futures cannot be maintained in uncritical manners.

The New Climatic Regime poses a serious predicament for mathematics education. Mathematics education as currently organized strives to fabricate Modern mathematical subjectivities, the types of people who, with a mathematical mind, can perform a Modern lifeform. It seems to me that the amplified desire for mathematical subjectivation, as previously described, needs to be questioned in a time when insisting on continuing to be Modern is a death sentence for the Earth —humans included. This provocative statement may generate questions and controversies in very many directions: Which mathematical contents should then be taught or need now to be included? Which pedagogies should we now practice so that school children start caring for the Earth? What is the mathematics education for a future… and which possible future? As previously argued, I try to resist the temptation of educationalizing climate change to be addressed with mathematics education. I prefer to take Latour's idea of a New Climatic Regime as a call to think and act seriously on the basic assumptions on which we have built the values and epistemic virtues mobilized in Modern mathematical subjectivation. Instead of gearing towards didacticizing, we need to turn to investigations in the philosophy and cultural politics of mathematics education to problematize and find new imaginations and orientations.

MATHEMATICS EDUCATION AND A TERRESTRIAL LIFE

The image of the Earth "from above" captured by the first astronauts became an iconic symbol of a modernized world and the encompassing views of nature and knowledge: the "Blue Planet", the Earth in harmony owned and controlled by humans and their technology. The New Climatic Regime invites to localize our relationship with that earth as our material conditions of existence "from below", within the critical zone of life (Latour & Weibel, 2020). This is the fragile membrane from the sedimented ground to some layers of the atmosphere on which existence as we know it has always unfolded. We need to land on earth (the one which, by the way, we have never left)! Latour (2018) proposes to talk about the *Terrestrial* as a concept that reminds us about the mud in the territory on which all beings of all species are connected in systems of subsistence. Contrary to the view of the Blue Planet or to the view of an abstracted Euclidean space or a Cartesian defined land on a two-dimensional map, the Terrestrial embraces the wide network of inter-dependent "animate beings —far away or nearby— whose presence has been determined —by investigation, by experience, by habit, by culture— to be indispensable to the survival of a terrestrial." (p. 95-96) Rather than talking of "humans" to name the inhabitants of the territory, the term *terrestrials* directs attention to all those mud-bounded beings whose chances of life and interconnected. The possibility of generating alternative descriptions in the Terrestrial, a new imagination, is at the core of a different political orientation in the New Climatic Regime. In the study of new ethics for organizations in facing climate change, Jørgensen (forthcoming) proposes to think Latour's invitation in terms of *story-making*. The terrestrials' capacity of re-birth and engendering is for Jørgensen central in political agency. Our question can be reformulated then as: Which new stories could we make to reimagine and reconfigure the basic assumptions of mathematics education for the Terrestrial?

A first possible reaction is to say that no other stories are necessary. What mathematics is cannot be changed. Its characteristics as a form of reason have made it such a powerful knowledge, tool and form of dealing with the world. And even if mathematics has been described in multiple ways (e.g., Ravn & Skovsmose, 2019), its contribution to the creation of a dominant, techno-scientific material culture is perceived as its power. That Modern mathematical subjectivation takes place —sufficiently— to feed and maintain such a culture and its systems of symbolic and material production grants its needed prominence in schools and societies. Rather, mathematics and school mathematics as we know them and in fully expanded power through better pedagogical inscription devices are to be defended. “Alternative story-making” could be left to other school subjects. School mathematics is and should be school mathematics. Period!

Second, an important caveat: Advocating for alternative descriptions or new story-making is not to be mistaken for a call to destroy objective facts and certainty in favour of a relativism where “anything goes” and that opens the doors to the type of anti-scientific “alternative facts” that pullulate in current controversies. To seriously start rethinking and making new stories, in my view, requires taking a critical stance to our well-intentioned desires for more and better mathematics education as we have it now: More of the same which in first place is implicated in leading us where we are will not constitute any viable alternative. It is to be critical to our hopeful impulses of educationalizing for change without touching on the fundamentals, as Labaree (2008) warned us.

The critical stance requires different interconnected fronts that together may help us make other stories of mathematical subjectivation. Alf Coles’ plenary in this conference offers concrete examples of *curricular and pedagogical work* that explore the possibilities of different types of school mathematics with a sensibility to the living beings in the territory and life of communities, as well as with a sensibility to other forms of mathematical work. The 2023 ICMI-sponsored on-line symposium “Mathematics Education and the Socio-Ecological” offered examples of similar initiatives in many countries around the world.

Even if the possibilities of story-making for alternative pedagogical imaginations may be more appealing to the large mathematics education community and its perceived need of a tight link to “practice in classrooms”, I hope that I had made clear why consistent work on a *philosophical* and *conceptual* front is paramount. Part of such front can revisit some of the Modern stories of mathematics that are connected with the characteristics of Modern mathematics subjectivation in schools, in search for elements to reconfigure ways of doing school mathematics. It requires investigating mathematics education beyond classrooms and schools and in conversation with a variety of current fields of study to displace and decenter narrow stories of the power of (school) mathematics.

Previously, I have pointed to some persistent old stories connected to Modern mathematical subjectivation that are problematic in face of the New Climatic

Regime (Valero, 2019). Here I have delved into the story that mathematics is universal and fabricates a subjectivity of a homeless mind. A power effect of this is the sense of self with non-location or a view point from nowhere. This effect extends to the view that teaching and learning practices are also detached from location or, in other words, seem to happen outside of the interrelations of the critical zone of life in the Terrestrial. Researchers have explored ways of conceptualizing the inseparability of place and space from the configuration of mathematics education practices. For example, Larnell and Bullock (2018) propose a framework that connects different spatial layers to explore how the concrete organization of resources in classrooms, schools and communities reproduce class and racial inequalities in mathematical learning and results.

Another of those stories is that mathematics resides in the mind. This has been substantially challenged in, for example, the work of Elizabeth de Freitas, Natalie Sinclair (2014) and collaborators who have teased the idea that mathematics is disembodied. The study of (school) mathematical practices has led them to trace the materiality of doing mathematics in the entanglement of body and tools in practices such as diagramming (de Freitas & Sinclair, 2012). In connecting studies of science and technology, with contemporary philosophy and performing moves of historization, they articulate an inclusive materialism that shows other “minor mathematics” (de Freitas & Sinclair, 2020) and how it is possible to engage in other forms of mathematical subjectivation. They put forward conceptual and empirical experimentation that open to, “affirm the creative power of abstract forms as alchemic expressions that are powerfully earthbound. Rather than blame abstraction for its detachment from the real, [they] show how abstract forms can be a source for thinking creatively about earthly spatial dynamics” (de Freitas et al., 2022, pp. 518-519). In this way, they examine “What is a mathematical concept” (de Freitas et al., 2017) in ways that challenge narrow understandings of mathematics, mathematics education and mathematics education research.

The sustained collaborations mentioned above are examples of the effort of many mathematics educators around the world (e.g., Le Roux et al., 2022; Rubel et al., 2021) to engage in the intellectual activity of thinking seriously what has been and could be mathematics education in a New Climatic Regime. Alternative ethical, political and aesthetical sensibilities can provide an orientation to navigate the multiple pressing issues of the moment. It is only with moving into unexplored territories in our field where we can make other timely stories of mathematical subjectivation for a Terrestrial life.

ACKNOWLEDGEMENTS

I thank the members of the SOCAME research group, RelMaS doctoral program and IMMPACT project for the collaboration that feeds my work. Without the broad network of colleagues in Brazil, Chile, Colombia, Sweden and USA, these ideas would not come to be. Special thanks to F. Beccuti, I. Lundegård, and H. van

Steenbrugge who discussed previous versions of this paper; and to W. van Dooren and L. Rubel for their comments as members of the PME 46 organization.

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RESEARCH FORUMS

PME-46

MATHEMATICS EDUCATION
FOR GLOBAL SUSTAINABILITY

HOW SOCIO-ECOLOGICAL ISSUES ARE URGING CHANGES IN CURRICULUM (AND BEYOND)

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INTRODUCTION

In this Research Forum, we investigate ways to expand the curriculum of: school mathematics, university mathematics, mathematics teacher education and community groups, drawing on interdisciplinary and transdisciplinary approaches. The aim of this Research Forum is to: (i) increase awareness of ecological and social crises, (ii) raise ecological, socio-economic and socio-political dilemmas and related pedagogical tensions, (iii) explore solutions that rely on mathematics, and support mathematical development, ownership, and responsibility. It has been claimed that "*everything we know about the world' climate—past, present, and future—we know through models*" (Edwards, 2010, p. xiv, emphasis original). Though we might dispute Edward's use of the word "everything", mathematics is fundamental to the way we come to *view* sustainability issues, as well as how we can *respond* to them (Barwell, 2013; Skovsmose, 1994). Mathematics creates a particular view of the nature of climate change (Skovsmose, 2021), and we believe this relationship needs to be held up for critical examination.

The Research Forum will be a gathering point for theoretical and empirical work that we are labelling "socio-ecological", a modifier that will be explored in the next section. In this document, we next outline some of the philosophical commitments of what the authors of this Research Forum have come to describe as a "socio-ecological" perspective or practice of mathematics education. We then describe methodological considerations, before offering examples of a range of curriculum innovations. We conclude by reflecting on the reported work and offering suggestions and prompts for future research. We have collaborated on the text below and, although responsibilities for different sections were distributed, we write with a collective voice. Author names at the top of this page are arranged alphabetically.

PHILOSOPHICAL COMMITMENTS OF SOCIO-ECOLOGICAL PERSPECTIVES

At its simplest, a socio-ecological perspective or practice implies that the social environment, the natural environment, and their intersection do not take place as a neutral backdrop to our concerns as mathematics education researchers (Coles, 2022). We, thus, first look at past work in psychology theorising the ecological and the social not as a fixed background.

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 71-102). PME 46.

Bronfenbrenner (1977) proposed a socio-ecological psychology conceptualised as a series of nested systems within which any individual is embedded. These were: a microsystem, of the immediate relations between an individual and its environment; a mesosystem of interrelations within significant settings for an individual (such as a school); an exosystem, extending the mesosystem to other social structures; a macrosystem, of the "general prototypes" (p.515), which exist in a society and constrain the relationships within the other systems (e.g., patterns of what school classrooms are like). Socio-ecological psychology has influenced the related fields of organisation theory (Boons, 2013) and ecological systems theory (Burns et al., 2015). However, central to the concerns of socio-ecological psychology and its offshoots has been an exploration of the behaviour of individuals - this can be observed in some of the visualisations of Bronfenbrenner's framework, which has the individual at the centre. Our use of socio-ecological intends to de-centre concerns away from individuals and, indeed, away from mathematics, towards socio-ecological issues and practices: in other words, we aim at considering not the individual per se, but in her continuous and complex relationship with the environment, both social and natural, at all the levels of microsystem, mesosystem, exosystem and macrosystem, as outlined by Bronfenbrenner (1977).

Given space constraints, it is not possible to do justice to the range of our influences and so we focus here on reviewing, briefly, philosophical ideas from: critical mathematics education; work on Indigenous ways of knowing; and, socio-political research. We view socio-ecological practice not as replacing any of the perspectives we now review, but rather as a call to bring the ecological side of the socio-ecological into sharper focus. Our review of these fields of work focuses particularly on what mathematics is, or becomes, from different perspectives. We draw out broader philosophical issues towards the end of this section.

Andersson and Barwell (2021) propose that critical mathematics education can be characterised as: "*driven by urgent, complex questions; ... inter-disciplinary; ... politically active and engaged; ... democratic; involv[ing] critique; and ... reflexive and self-aware*" (p.3, italics in original). Socio-ecological practices potentially encompass all these elements and hence, as reported in Boylan and Coles (2017), it is surprising that, going back 8 years and 15 years, in two edited collections of critical mathematics education papers (Ernest, Sriraman & Ernest 2015; Alrø, Ravn & Valero 2010) only one of 34 chapters specifically relates to an ecological crisis (D'Ambrosio, 2010). However, this situation is beginning to change. Skovsmose (2021) identified three relationships between mathematics and crisis (including ecological crisis): (i) mathematics can "picture" a crisis, for example through classical modelling, (ii) mathematics can "constitute" a crisis, for instance, when models affect action in real time, leading to runaway behaviour (financial crashes are examples of mathematics constituting a crisis), (iii) mathematics can "format" a crisis, which is:

> where a mathematical reading of a crisis brings about a way of acting in the critical situation. It might turn out that these actions function adequately and that the crisis comes under control. The opposite might also be the case: the reading might be a misreading, and the actions might be counterproductive. In both cases, the mathematics formats both readings and actions, and in the end, it formats the dynamics of the crisis itself. (2021, p.370)

Skovsmose uses climate change as the example of mathematics formatting a crisis. Drawing on Barwell (2013), he argues that mathematical models of the climate do not stop at description and prediction. Models become a key to the way in which we think and act in relation to the climate - the models are "performative" (Skovsmose 2021, p. 378) as well as descriptive and authoritative. Climate models provoke action and change in the very things they model, and are far from passive representations of reality. Skovsmose's (2021) typology points to possibilities for classroom practices of mathematics and has some overlaps with Renert's (2011) categorisation of approaches to questions of sustainability. These approaches are: accommodation (i.e., sustainability is brought in as context for mathematics, little else changes), reformation (sustainability is beyond a neutral context and education connects to outside communities, but overall structures are not questioned), transformation, where mathematics education is re-shaped towards activism and knowledge viewed as "approximate, relational and provisional" (Renert, 2011, p.22).

Insights from socio-political perspectives on mathematics education (Pais & Valero, 2012; Gutiérrez, 2013) are highly significant to socio-ecological practice, in our view. Indeed, when we write "socio-ecological" we are hearing the "socio" side of that as meaning socio-political. In particular, when it comes to questions of climate change (which might be seen as a socio-ecological issue) our concerns point to questions of justice: climate justice, environmental justice, racial justice, gender justice, spatial justice. Socio-political research views mathematics education within its historical, social and political lineage and points to blind spots, such as who gets marginalised through practices which may appear "neutral". Through habit, as mathematics educators, we can become desensitised to issues around how power, knowledge and subjectivity are assigned and used, in the classroom and beyond. Socio-political perspectives overlap with critical mathematics education and, indeed, Andersson and Barwell (2021) highlight a Foucauldian School (of critical mathematics education) which aims also to highlight questions of power and disrupt unjust hierarchies (e.g., Walkerdine, 1990; Walshaw, 2007).

Thirdly (in terms of reviewing work from which we draw inspiration), in relating to ethnomathematics and Indigenous ways of knowing, we foreground Indigenous perspectives to this discussion about framings of the socio-ecological. Gutiérrez (2017), for example, proposes the term mathematx to point to alternative possible practices of mathematics education. Drawing on Indigenous epistemology, ontology and values, Gutiérrez considers how

> mathematics education can support interdependence (‘In Lak'ech’ in Mayan), indeterminacy (‘Nepantla’ in Nahautl) and reciprocity. Gutiérrez argues for an approach of pluralism, inclusive of different ways of knowing (Santos, 2007). Gutiérrez is careful to point out there is no single Indigenous worldview, while also noting commonalities around the conceptualisation of a relational ontology that disrupts binaries such as self/environment, living/non-living, human/non-human.

Nicol et al. (2013) illustrated the potential for culturally responsive mathematics education, as a way of making mathematics teaching more relevant to Indigenous communities and peoples, and in a manner that benefits all. In the context of the learning of mathematics teachers, Nicol et al. (2020) discuss their work on de-colonising educational practices, to re-centre Indigenous perspectives and community needs, within colonial institutional structures. Across this work (in classrooms and with teachers, including how the research is reported) there is a feature of the importance and power of storying (this is discussed in more detail in the next section). Again, there is a strand of practice here that disrupts the postulation of a stable identity and subjectivity against a fixed outside or environment.

The authors of this Research Forum recognise the importance of dialogue, uncertainty, risk, openness, self-awareness, which is raised by the work reviewed above. A socio-ecological practice entails a commitment to an on-going interrogation of experience. We recognise how easy it is to fall back into assumptions, such as taking the world around us as fixed, or taking institutional structures as neutral. From a socio-ecological perspective, there is no physical world, which acts as a stable environment, against which living beings adapt, evolve and respond; rather, for all of us, our environment is other living beings, and their remains, with which we are inextricably connected.

We raise a question of ethics and propose the significance of a dialogical ethics (Barwell et al., 2023). A dialogic ethics troubles the self/other binary. Rather than viewing ethics as prompting consideration of the actions of an already constituted self, from a dialogic perspective (and drawing particularly on Levinas, 2011 and Bakhtin, 1993) it is through recognition by an “other” that an individual comes to recognise her own self as a subject capable of ethical responsibility. It is out of her relationships that an individual is able to individuate, and it is out of her relationships that she becomes responsible and answerable to others (in all their human and non-human richness and diversity). In keeping with the socio-ecological practices sketched above, a dialogic ethics shakes up a human centring of concerns and may help attune us to processes of individuation and the ethical relationships which run alongside.

Having reviewed some of the philosophical commitments of a socio-ecological practice, including the significant strands of aligned research from which we draw, we now move on to consider methodological questions. In particular, we consider what methodological approaches are consistent or appropriate for working with socio-ecological issues. Of course, methodological considerations interact with philosophical ones and there will be overlaps across this section and the next, particularly in relation to Indigenous ways of knowing.

METHODOLOGICAL APPROACHES TO SOCIO-ECOLOGICAL ISSUES

In this section, we examine several methodological approaches used to study socio-ecological issues in mathematics education. We acknowledge that there is a broad range of studies in this area, with a diversity of methodological approaches. However, we identify a common theme across these studies: that of a focus on seeking out and drawing on the voices of multiple participants. We begin this section by examining methodological approaches used in several studies that focus on centering minoritized and marginalized groups, with an emphasis on those that centre Indigenous knowledge and decolonization in mathematics education. We end this section by examining methodological approaches used with mathematics educators, in studies with a focus on socio-ecological issues.

Environmental crises, including access to drinking water; cycles of droughts, flooding, and rising ocean levels; air quality, rising temperatures, and access to shade, significantly (and usually unevenly) impact Indigenous communities, as well as those in marginalized and minoritized communities. Historically, Indigenous knowledge systems have been fundamental to survival and flourishing of people and wider society as they draw upon holistic, multi-disciplinary systems that underpin practices related to food production, health, education, and conservation (Bianchi, 2018). Yet in school settings and perhaps mathematics classrooms in particular, Indigenous knowledge systems are frequently ignored, neglected, and devalued. Often, there is little known about everyday community practices connected to mathematics or how families interact in ways related to mathematics in out-of-school settings (Civil, 2016; Mills et al. 2019). This is a key equity issue: in relation to developing knowledge of sustainable Indigenous ways of being and in reflecting on curriculum innovation that positions marginalised communities as knowledgeable and as thinking and being in mathematical ways.

Developing understanding of Indigenous or other local knowledge systems requires research studies that position students, families, and communities as experts who can share their knowledge with educators. Participatory research methodologies are built on principles of inclusion and offer insight into often unrecognised knowledge. There are varying forms of participatory research, however, frequently, research studies in mathematics education frequently engage story-telling or narrative inquiry as an inclusive participatory methodology. For example, studies (e.g., Garcia-Olp et al. 2022; Nicol et al. 2020; Ruef et al. 2020) from the North American region including Canada and the United States of America, utilise methodological designs of collaborative story-telling groups to connect to land and place-based pedagogies with a focus on local issues that in turn connects to wider global issues. Nicol and colleagues (2020) developed a structure for their collaborative research group: living and telling stories which introduce and provide insight into the positionality of each group member, re-telling stories in response to shared texts and re-living as educators and researchers continue their work in changed ways from listening and reflecting upon stories.

A key aspect of participatory research and narrative inquiry is developing respectful relationships with local communities. In the case of Indigenous peoples, this means attending to understanding of ancestral knowledge through listening to the stories of Elders and centring these in mathematics classrooms. Often these stories have direct connections to socio-ecological issues, for example, Nicol et al. (2020, p.196) share the insights of a Haida Elder GwaaGanad Diane Brown who "spoke of the need to treat the ocean as our relative. She spoke of supernatural beings who could shape-shift between worlds reminding us of the need for, and consequence of, disrespecting the gift of food." In this way, stories were shared that focused on relationships with the ocean and this opened up consideration of the possibilities of learning that the ocean offered. Other studies such as that by Gracia-Olp and colleagues (2022) involved connecting Indigenous youth with Indigenous elders to understand mathematics and science concepts focused on sustainability including the building of tipi.

Methods of participatory research and narrative inquiry are applicable in urban contexts as well, and are especially important relative to marginalized people in cities (which can include Indigenous peoples). A "right to the city" implies participation in the production of urban space, freedom from imposed segregation, the right to public services, and access to the goods of city life (Lefebvre, 1968). However, any city can no longer be viewed independent of its changing global context and especially, dwindling environmental resources. Marginalized people in cities suffer unequal access to vital resources, be it drinking water and increasingly important, shade from the sun. Mathematics can be used to quantify that unequal access and to advocate for justice and equality. Using mathematics to analyze unequal distributions of resources often culminates in attempts to raise individual awareness and thereby blame the victims by emphasizing individual choice, rather than identifying and transforming systemic injustices that shape those unequal distributions. Examples of projects that collaborated with marginalised youth to use mathematics to identify unequal distribution of sustainability resources in urban spaces and reimagine those resources include data-informed redesigns of public parks (van Wart, Tsai & Parikh, 2010) and bicycle paths (Taylor & Hall, 2013).

A related participatory research approach that has been used to generate stories from young people and their communities is photo-voice methodology. Beyond mathematics education, this has been successfully used with participants from marginalised communities to support them to identify, document, and represent the strengths of their community (Lienbenberg, 2018). This method originates from community and health initiatives but is widely used in projects related to education, empowerment, and social change. For example, two research studies (Hunter, 2022; Hunter & Restani, 2021) used photo-voice and photo elicitation interviews to explore mathematics of Indigenous Pacific people living in the small island nations across Moana-nui-a-kiwa (the great Pacific Ocean) and Aotearoa/New Zealand. This methodology proved effective at providing a voice for young Pacific people and an opportunity for them to share their ways of being and knowing related to mathematics through both taking photographs and telling the stories of the images that they captured. Evident in the story-telling related to the mathematics of Indigenous Pacific families in Niue was the interwoven elements of science, conservation, and sustainability to maintain life in remote island nations. Traditionally, in this Pacific nation, niu (coconuts) take an important role with the plant material being used as both a food source and for construction. Hunter (2022, p. 195) provides an example of a student story of collecting niu:

> Niu, the coconuts, I'd have to teka (hook for coconuts) them and I have to collect them…husking it was difficult because if they were young coconuts and you'd husk them, you have to husk them carefully in case one would pop. Then I use the husk for compost, put it around the tree and I think it was around about five husks on one coconut, that's the average but if the coconut was really big I guess it would be seven depending on the width of each husk. It doesn't matter how many you have as long as it covers the stem of the tree.

In this example, we see how the participatory methodology of photo-voice opened up an opportunity for those who are seldom heard in mathematics education, Indigenous youth, to share their experiences which incorporate mathematics, sustainability, and conservation.

A third example of a participatory research method involves the techniques of participatory mapping and especially, that of counter-mapping. Participatory mapping indicates a shared authorship of a map. Counter-mapping signifies a map from the perspective of a group that is typically marginalised. For example, Open Street Map (www.openstreetmap.org) and Pollution.org are participant sourced alternatives that create city maps and track global air pollution and contamination respectively. Along with the map representations of participant sourced data are the data itself. Rubel and colleagues (2017) designed and studied the use of an interactive, digital mapping portal that visualized quantitative datasets and enabled youth to gather counter-stories in the form of audio interviews or photographs and position and link to those stories on the digital maps.

Both mathematics educators, and researchers working in research and practice-based studies related to socio-ecological contexts, recognise the need for methodologies that position participants and the wider community to think differently about mathematics (Li & Tsai, 2022; Yaro et al., 2020). As highlighted in the earlier part of this section, a first step in this process is drawing on the voices of multiple participants including Indigenous Elders and community members. Widening from this, there is also a critical need to position teachers including in-service teachers and pre-service teachers, to reflect on their understanding of mathematics and local issues and to view mathematics education as involving contextual understanding, critical thinking, modelling, and problem-solving (Helliwell & Ng, 2022; Li & Tsai, 2022). To achieve this, research studies frequently utilise methodologies that engage teachers in task design and/or implementation within the classroom. When working in Indigenous spaces, a key concern here is the development of mathematical tasks that are both accountable and responsive to place but that do not appropriate or trivialise Indigenous cultures and ways of being (Nicol et al., 2020). Yaro and colleagues (2020) use examples from Ghana to highlight how the development of mathematical tasks that draw on authentic contexts and local knowledge have the potential to position mathematics teachers to explore local issues and sustainability. However, a common theme across studies involving task design and implementation to focus on socio-ecological issues is teacher discomfort. For example, Yaro and colleagues (2020) reported teacher perceptions that the tasks were outside of school mathematics. This was also noted by Nicol et al. (2020), along with a secondary concern from teachers, that they may be appropriating Indigenous knowledge or not teaching tasks correctly aligned with Indigenous perspectives.

One challenge that is common across studies involving task design and implementation with a focus on socio-ecological issues is teacher discomfort. For example, Yaro and colleagues (2020) reported teacher perceptions that the tasks fall outside the perceived boundary of school mathematics. This was also noted by Nicol et al (2020), along with a secondary concern from teachers that they may be appropriating Indigenous knowledge or not teaching tasks correctly aligned with Indigenous perspectives. Another challenge is the inherent complexity of the socio-ecological. For example, Rubel and Nicol (2020) point to the example of the Hunter's Point Shipyard in the United States, a former Navy yard in a large city that despite (or perhaps because of) environmental hazards, evolved into an African American community. Current methods of cleaning these environmental hazards require their transfer elsewhere. In this case, the financial desperation of the Skull Valley Goshute Native Americans led them to accede to house those hazards in their less populated desert Land. However, the white residents in a nearby city ultimately blocked this transfer, by invoking environmental justice protections, with some irony, since these exist to protect marginalised peoples. Thus, it is not merely teacher discomfort; there are inherent complexities to socio-ecological issues, precisely because of layered inter-relations between and among humans, nonhumans, and the planet.

Having reviewed philosophical commitments linked to socio-ecological practices, and then methodological considerations, we now turn to examples of curriculum innovation. There are potential implications of adopting socio-ecological practices in mathematics education across all its fields of interest; we have chosen to focus on curriculum innovation since this is an area in which socio-ecological concerns can influence both content and approach.

CURRICULUM INNOVATIONS SHAPED BY THE SOCIO-ECOLOGICAL

In this section, we lay out work on curriculum innovation which can be seen as involving socio-ecological practices, drawing on the experiences of the Research Forum organisers. We first review briefly the broader field of curriculum innovation and the socio-ecological, and we then move into a more detailed expiration of past and current curriculum research on: school and community innovations; undergraduate innovations; teacher education innovations.

Review of past innovations addressing the socio-ecological

We first recall that this Research Forum builds on work that has been conducted over the last 10 years, linking the concerns of mathematics education to issues arising from environmental degradation and the socio-political implications. And yet, we recognise that what we are describing is a research area that gains little sustained attention. A forthcoming ICMI Study Volume on Curriculum Reform (2023) mentions “environmental disasters” in the introduction as one of the pressures on countries to engage in reform and Shimizu and Vithal (2023) point out the importance of school mathematics reforms dealing with:

> the global changes taking place in societies, as they confront different challenges of growing inequality, unemployment, poverty, mass migration, environmental disasters, various form of discrimination and conflicts (to name but a few) (p. 18).

And yet, throughout the whole Study there is little further mention of such environmental issues and their relationship to curriculum reform. One chapter raises the issue of global crises - but to point out its relative absence:

> We want, finally, to raise an issue of pressing importance which, however, does not appear in any of the documents presented at the ICMI Study Conference, that is the issue of when and how curriculum reform will pay attention to the global crises of climate change, biodiversity loss, mass migration, access to water and other worldwide issues. (Coles et al., 2023, p. 262)

Our overall impression, therefore, is that past efforts around mathematics curriculum innovation have paid little attention to questions of the socio-ecological. Various studies, especially outside mathematics education (e.g. Heo & Muralidharan, 2019; Grima, Leal Filho & Pace, 2010) highlight that, the need to preserve the environment has increased in recent years, given the continuous increase of wasteful consumption of resources. Hence, it is our view that schools, as creative members of the community, have a responsibility to take a stand, in terms of encouraging awareness, challenge and involvement in socio-ecological issues.

From the Organisation for Economic Co-operation and Development (OECD) perspective, Taguma et al. (2023) present common challenges that OECD countries face when redesigning their mathematics curriculum and how these challenges can be addressed. They introduce key concepts of the “OECD Learning

Compass 2030", a learning framework that describes the competencies students will need to face the social, economic, and environmental challenges. Taguma et al. (2023) point out that dealing with:

> environmental issues– population growth, wastefulness, resource scarcity, air and water pollution, and electrical energy demand – requires mathematical competencies, such as knowing how to solve mathematics problems involving basic computations, percentages, ratios, tables, circle charts and graphs" (p. 396).

These authors discuss some implications for designing mathematics curricula based on principles that focus on developing students' agency.

Despite the lack of attention to socio-ecological issues in past mathematics education curriculum reform, there are other strands of related curriculum research, which do tackle such issues and questions, some of which are touched on in the three sections that follow. These strands include work within STEM education, environmental education, work on inter-disciplinary and trans-disciplinary projects and work of socio-scientific problems. Interestingly, one commonality we have found in inter-disciplinary (Savard et al. 2017) and trans-disciplinary work (Solares et al. 2022) in primary schools, is a suggestion that there can be a reluctance from teachers to tackle mathematics. In both projects (Savard et al. 2017; Solares et al. 2022), there were opportunities for drawing out mathematical curriculum connections, within a broader task that students were engaged with, and in both cases these opportunities were either not taken up at all or taken up in a relatively limited manner. There are important questions raised, therefore, about the kinds of support that teachers might need to draw mathematical connections to inter- and trans-disciplinary problems and projects.

Curriculum innovation involving inter-disciplinary work potentially touches on both *what* is being taught and *how* it is being taught (Coles, 2023). In contexts where opportunities for changing the "what" of the curriculum are limited, it may be possible to develop work on the "how", for instance, supporting mathematical competencies, within tasks that do not have an explicit ecological dimension.

Also of relevance to thinking about past work on curriculum reform, is work done on mathematics education and social justice. In discussing approaches to social justice, Boylan (2017) identifies three "orientations", which are relevant also to thinking about the socio-ecological and mathematics education. These orientations are:

- Conservative: "improving learning opportunities and outcomes for those who experience injustice in education … based on teaching about the other" (p.370)
- Socially-liberal: "recognises that educational structures and practices need transforming … that teaching should be culturally sensitive and multiculturally competent … and that othering and privilege should be challenged" (p.370)

- Critical: "for social justice to be realised in schools, social structures beyond school need transforming … sociopolitical contexts need to be recognised and teaching needs to be a counter-hegemonic practice … education, thus, seeks to change students and society" (p.370)

Boylan uses these orientations to consider what it means to engage in "critical teacher education". Moving through the orientations gives a sense of practices with an increasing reach and ambition, in terms of change, from changes in the classroom (conservative), to changes in schooling structures (socially-liberal) to changes in society (critical). Boylan also suggests that ethics, more than justice, may be a productive source of action for a critical teacher/teacher educator.

While much has already been studied in terms of education innovation research that is geared towards transforming classroom practices (Cohen & Ball, 2007), very little research has been dedicated towards situating the classroom as part of its local community. In their recent investigation of sustainability-driven curricula, Makramalla and Tilley (2022) assert that questions of sustainability seem to be considered alien to the mathematics classroom in many places around the world. The classroom setup - as it stands today - remains isolated from the precarious reality of the community, where it is embedded. Hence, there is a need to re-envision curricular innovation to go beyond tools and competencies that make the classroom experience more creative. To that end, Mitchell (2021) has recently framed the integrated curriculum framework, a framework that embeds socio-environmental responsibility as part of the day-to-day classroom experience. According to Mitchell (2021), the learning outcomes of a mathematics curriculum need to take the form of practical tools with which to face the complex daily challenges that have been caused by climate change today.

Makramalla (8th author), with colleagues, has been developing a mathematics curriculum that is underpinned by case studies, which are based on climate change challenges around the world. The aim is that, as students engage with these challenges, they firstly develop a sense of awareness and responsibility towards the environmental challenges that surround them. This sense of responsibility is then transformed into an empowering sense of agency as students re-envision mathematics as the tool to combat these challenges. This is one of many examples of how innovations in mathematics education can lead to a re-envisioned perspective of mathematics as a subject matter and a transformation of re-envisioning oneself as a learner from being a passive recipient of knowledge towards becoming an active agent of change. This transformation of change agency within the student body is impossible if it is not fully supported by a transformation in the teacher's own self-envisioning (Makramalla & Stylianides, 2021). In other words, innovation in mathematics education needs to go beyond equipping teachers with more innovative approaches to teaching mathematics. It needs to address teachers' sense of agency to transform their roles beyond its situatedness in the classroom. We offer examples below which give some detail about what such work can look like in practice, first in school/community settings and then in the context of undergraduate mathematics. As has been surfaced already, and will again be apparent in the work below, a significant challenge for the mathematics teacher education community is to understand how mathematics teachers can be supported in reimagining their roles as practitioners and their classroom practices in relation to socio-ecological issues. We address questions of innovation in teacher education in the third section that follows.

In dealing with the three areas of (1) school and community innovations, (2) undergraduate innovations, (3) teacher education innovations, we are not suggesting these areas cover all curriculum innovations, rather they cover a wide range of ages and aspects of mathematics education and also correspond with the expertise of the Research Forum organisers.

School and community innovations

In this section, two examples of school curriculum and community innovation are showcased. The first example comes from an activity based on the case of the Sumas Prairie flooding. Sumas Prairie lies between Abbotsford and Chilliwack, British Columbia. The area is approximately 100 kilometres east of Vancouver and is a very large fertile area with many homes and farms. Much of the dairy and poultry in grocery stores in Vancouver come from the area. In November 2021, devastating floods hit the area causing significant damage to homes, farms, livestock lives lost and road closures. One of Vancouver's main highways passes through the area; it was blocked by the flood.

Data about a curriculum innovation has been collected from a Pre-calculus grade 11 class in a high school in Vancouver, BC, Canada. Students were taught a lesson by the teacher that was developed with Chorney (Research Forum 3rd author). The lesson included the storytelling by the teacher, the exploration of Google Mymaps and discussion between students based on what they noticed. The students were then given a worksheet and the data, some of which will be shared below, comes from the worksheet. The overall goal of the research is to bring mathematics into the conversation of socio-ecological concern and to explore how mathematics is used in reasoning and how it can influence personal meaning making.

As a premise, we maintain that mathematical models are not the only way we come to know about climate change, in fact, mathematical models can conceal inequalities that result from the application of these models (section on Philosophical Commitments, above). There is, therefore, a need to consider ways in which mathematical modelling can be combined with the imaginative engagements and qualitative tools of the social sciences and humanities as a way of addressing the socio-ecological issues of today's world.

The Canadian government and news media announced that the flooding was due to atmospheric rain. The rain during that month was unprecedented and was a big cause of the flood but there was another reason contributing to the flood; the area had previously been a lake that the Sumas Nation lived on, and with, for over 1000 years. For the Sto:lo people (River people), 85% of their diet came from the lake and there is evidence that Indigenous people used the lake as far back as 400 BCE. But in 1924 the lake was drained. The rivers feeding into the area were redirected, specifically the Chilliwack river, and a pump station was put in so that the Sumas area could be used as farmland. There is evidence that the government drained the lake without any consultation with the Sto:lo people and that they drained the lake for purely economic purposes. One Sto:lo person said of the 1924 drainage “They choked the lake”. The challenge now is that with the changing climate the current infrastructure is not capable of keeping water out of the area if another heavy rain occurs.

Chorney shared this story with many teachers. One teacher, in particular, who teaches in a high school in Vancouver, has worked with him to develop a lesson for her students. The interest was to mathematise the lake, particularly in relation to its size; it was approximately 40 square kilometres. The students use Google Mymaps to draw a boundary around the lake as it once was pre-1924 (there were images shown to the students). Once the boundary is drawn, Mymaps calculates the perimeter and area but the students are asked to calculate the volume of the lake. The students were also asked to drag the bounded area to another part of the BC map, asking them to compare the size of Lake Sumas with a landmark the students are more familiar with. The purpose of the lesson is to bring attention to the size of the lake and to recognize that it was a significant sized lake and therefore caused a profound change to the local environment.

From a research perspective, Chorney was interested in the reasoning of students, in particular how students frame their ethical understanding of the Lake Sumas devastation and how they make explicit characteristics of the devastation in terms of the political, economic and the social. As it was a mathematics class, specific interest was how students framed these characteristics using mathematics.

The theoretical focus of this research includes two strands. In one strand, researchers asked how students reason using mathematics calculations as justification. Using the measurements of Lake Sumas, for example, Chorney investigated how students create arguments and rationalisations based on those metrics. Building on socio scientific practices (Sadler et al., 2007), highlights how students navigate their development of linking mathematics with the social ecological world as they become aware of characteristics such as complexity, inquiry, scepticism, perspective taking and ethical considerations. One student wrote:

> allow[ing] the lake to reform, as obviously that is also what Mother Nature would have wanted … If, in 1924, Sumas Lake could hold that much water, with the more recent atmospheric rivers getting bigger and badder and the land being below sea level, more drastic and severe flooding in the area is basically inevitable.

This student is conveying a perspective that the lake will return and bases their argument partially on their calculation of volume.

The other strand of research is based on resemiotization (Iedema, 2001; 2003). "Resemiotization is about how meaning making shifts from context to context, from practice to practice, or from stage of a practice to the next" (Iedema, 2003, p. 41). Many students drew on their own experience to give meaning to the Lake Sumas draining, not in terms of the draining itself but in terms of connecting personally with the idea of familiarity, loss, and personal connection. For example, one student, in noticing how big the lake actually was, wrote:

> I definitely think size should always be a consideration with large decisions. This reminds me of how my family in Japan told me that there's actually a lot of earthquakes, but they're so minor that a human isn't bothered by it. So, the size of the disruption is always important and is a way to understand if action should be taken.

It seems important to recall that "in many cases, resemiotization involves introducing new semiotic resources, and may result in metaphorical expansions of meaning as functional elements in one semiotic resource are realised using another semiotic resource" (Iedema, 2003, p. 12). When students construct a boundary around Lake Sumas and drag that bounded object over familiar landscape, as the student cited above did, a new semiotic resource is introduced making an opportunity for a multimodal assemblage.

This personal connection is a form of intersemiosis (Jewitt, 2009) because they are drawing on their own experience and introducing a new sign, for example, that of recollecting stories of earthquakes in Japan.

In a similar way, drawing on the perspective of sustainability as a resource of reason, one student writes:

> I do believe the volume and area should be taken into consideration and draining of the river [sic] as the extent of the flooding in area and volume possess significant danger to both the livelyhoods [sic] and safety of those in the area, as well as those affected by disruptions in supply chains. But I maintain that sustainability should be at the forefront of the resolution is fixing an issue caused by environmental degradation with an unsustainable method will only continue to pose future threats.

How students develop their own stance on such social issues as the Lake Sumas story can depend both on reasoning and/or intersemiosis. A particular interest in this research is how students tell their own story to give significance to the issue. Mathematics, in this case, is not just being pointed at, but is being personalised. The mathematics was not just calculations but also the expression of students' own personal understandings of what that mathematics means and also how it was used as a justification. In this research, mathematics is not a detached, objective knowledge that is inserted into reasoning. Mathematics is integrated with a personal component of developing a sense of meaning and story, related to the social issue.

In terms of the more explicit calculations, new methods to calculate the depth of the lake were introduced by students, one which included dissecting the lake into 10 sections and measuring the middle of each section and averaging overall. There were challenges finding volume since the surface area was approximately 40 kilometres squared but the depth was approximately 3 metres so changing units to calculate volume was necessary. In comparing the outline of Lake Sumas with their own personal landmark, many students dissected Lake Sumas into as many parts as their landmark fit, to determine the proportion.

In making distant historical events not only tangible but framed more personally; we see mathematics as a way of doing this. Of significance is that the story being created is not directly about Lake Sumas but rather as a way of creating a personal experience in alignment with the Lake Sumas story. Connecting to environmental issues, to Indigenous issues, historical events, is challenging to make personal as it may be from a different time and involve other people. Engaging in this mathematical activity, forging connections to the socio-ecological concern, students are able to connect to something quantitative, something concrete, so as to legitimise to a greater degree the story of Lake Sumas and the ongoing ecological concerns.

The second example of a school-based innovation that we offer in this section, we will describe more briefly, as it was the focus of a research report at PME 45 (Coles et al., 2022). In this example, a network of researchers, teacher educators, scientists, teachers and community activists was established to address the question of how the primary school curriculum in Mexico can adapt to take account of local and pressing socio-ecological issues.

In particular, the network was focused on the Atoyac River basin, South of Mexico City. The Atoyac River is the third most polluted in the country and the region suffers from increased rates of childhood leukemia, spontaneous abortions, cancer, among other impacts. The inspiration for the network came from the scientists working in the region who recognised that the centralised curriculum in Mexico precluded a focus on this devastating socio-ecological event, happening on the doorstep of many schools. The issue noted in the Lake Sumas work was a key driver for the work in Mexico also – how to connect with environmental issues. As Solares et al. (2022) report, one issue with environmental damage is that, when it has been on-going for decades (as in the case of the Atoyac River pollution) then young children growing up have never known anything different.

In the Atoyac River project, the network decided on a structure of supporting one primary school to develop a "Memorial Museum" to the river (Solares et al., 2022). This museum had three galleries, one gallery looking to the past and aiming to recapture some of the oral histories of what a healthier Atoyac River was like. A second gallery focused on the present and the current (and fluctuating) levels of pollution. A third gallery looked to the future and what might be the route back to a living river. This museum was created physically and replicated in a virtual environment (see, https://red-comunidadcienciaeducacion.org). The physical museum was made to be transportable and taken to several community centres in the Atoyac Region.

The project showed that, even in the context of a centralised and mandated curriculum, that innovation is possible. Teacher were creative, in linking text-book tasks with the Museum project, although, as stated earlier, this was only rarely done with mathematics (Solares et al., 2022). The museum project was intensive on the time of network members (partly as the project occurred through the Covid years). The network is currently developing ways to scale the work which is not so labour intensive.

Looking across the two school innovations reported above, we observe the potential for mathematics to become a school subject that is experienced as personal, integrated, meaningful and linked to activism. We also note the institutional demands and complexities which may work against enacting such an image and the need for support for teachers wanting to develop and innovate in such ways (see section on "Teacher education innovations", below).

Undergraduate innovations

In this section, an example of university mathematics curriculum innovation is presented and it comes from an undergraduate course in Environmental Sciences. We firstly highlight that undergraduate students enrolled in these courses face, more than others, topics that are under focus in many scientific and political debates at various levels all over the world, but that are also tackled in much less informed contexts such as mass media and social media, where fake news are massive and pervasive. Oversimplified models of reality are often proposed in these contexts and it becomes necessary to both master and apply mathematical models in order to correctly interpret the relevance of the data collected about socio-environmental phenomena, on one side, and on the other side scientific knowledge to understand the extent of the issues at stake, and their inter-dependencies. When people talk about climate change they mainly draw information from TV, newspapers, internet and school, but the contents of coverage differs considerably country to country. An observation from 8 years ago was that: "German media display rather 'warmist' standpoints similar to the 'anthropogenic climate change as a global problem' frame, whereas media coverage in the US and Australia is stronger polarised between sceptics and warmists who devote more attention to the 'scientific uncertainty' frame." (Schafer, 2015, p.6). Gregersen et al. (2020) found that political orientation alters concern about climate change. Thus, causes and consequences, confirmed by science and data, become a matter of opinion and, on the other hand, communication becomes more effective if it is supported by political elites and advocacy groups, even if they are alien to both research and science.

Heo and Muralidharan (2019) recall that environmental knowledge, concern about the environment and perceived effectiveness of one's own actions, are the variables that mostly affect individuals' sustainable choices. Grima et al.'s (2010) study points out a rather more complex scenario, as they investigate undergraduate Environmental Sciences students' knowledge and perceived responsibility about environmental issues, revealing that even these students hold conceptions that are inconsistent with predominant scientific understandings: factual misconceptions arise from inaccurate or incomplete information to which students are exposed (Huges et al., 2013), and exploring misconceptions through real data from environmental control institutions proves to be useful in order to break student biases about climate change (Rosling, 2018).

The activity that Rosling proposes allows misconceptions to emerge and compares them with real, correct data represented in graphs. Students should be able to derive data directly from the source, without any mediation by mass media. The importance of visualisation is crucial for this activity: students are required to read graphs and to communicate what they infer with other people (Franconeri et al., 2021).

In the research connected with these ideas, the first, second and fifth authors of this Research Forum wondered what would help students to perceive a chart as more transparent, i.e. the quality of showing ‘hidden’ data inside itself, like: the slope of the curve, the relationships between variables, the variation with time. In the hypothesis of improving the engagement of an observer who is analysing a graph, the authors thought that a better aesthetic could increase the time of observation and especially the pleasure of relating to it. To test this assumption a presentation was shown to a class of students in the first year of an undergraduate course in Environmental Sciences: it consisted of four different graphs for the same chemical component in water pollution of textile factories, in the territory of Biella, Italy. The difference in style and method of construction among the four graphs presented was intended to emphasise the different aesthetical features of graphs representing the same data. In Figure 1, a 3D vertical bar chart, a radar chart and a line chart can be seen as examples. Data collection consisted of a multiple-choice questionnaire made up of 15 questions (one for each chemical) related both to aesthetics and transparency and to the effective understanding of the graphs shown.

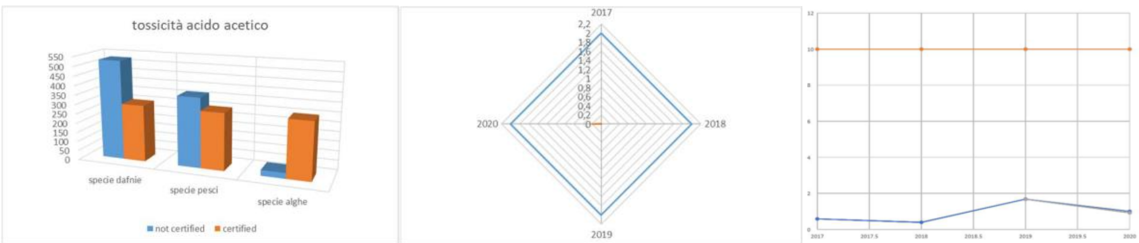

Figure 1, left: water toxicity of acetic acid; middle: time series from 2017 to 2020 of chrome emissions of the fabric (orange line) compared to European standards (blue line); right: time series for nitrogen emissions of the fabric (blue line) compared to European standards (orange one).

In general, the study found that the highest level of consistency (i.e., the percentage of students considering the prettiest graph as the most transparent) is achieved when there is a balance between transparency and aesthetics: the beauty of a graph alone does not guarantee that it is also perceived as transparent, as well as a transparent chart does not ensure that it is also pleasant. In particular, two tendencies emerged from students: it seems that those students who experience difficulties in mathematics see aesthetics and transparency as closely related, but it is not clear whether it is the perceived beauty that makes the graph appear as clear, or if it is its perceived comprehensibility that makes it feel beautiful. All in all, those who have a weaker mathematical knowledge seem to be less able to separate aesthetics from understandability, and to comment differently on each of them. In those who have well-developed mathematical skills, conversely, a separation has emerged between aesthetics and transparency: comprehensibility and beauty do not always coincide.

We can notice that mathematics plays a foreground role in the activities described above. We now showcase an example of another kind of activity, based on a learning game, in which "the formatting power of mathematics" (Skovmose, 2015) is limited and socio-ecological dilemmas emerge. We now recall game-based learning very briefly, then we introduce the game.

Learning games are meant for teaching skills, knowledge and attitudes rather than just entertaining (like in a game): in their review of research on learning games, Pan, Ke and Xu (2022) report that, unfortunately, the huge majority of games are employed for involving students' low-order cognitive skills. One exception is, in Pan et al.'s (2022) review, the simulation game genre, which is a way to simulate a complex situation and was found to stimulate high-order thinking (Pan et al., 2022). Pointing to educational proposals that allow for a revision of undergraduate curricula, this represents for us a source of inspiration. Furthermore, the aforementioned review notes that the individual play mode is more frequent than the collaboration one. Thus, in what follows we present a game-based learning activity that tries to overcome this limitation.

The game starts with a short introduction about the case of Carl Pepper, a farmer in Texas who changed his cotton cultivation mode from the conventional to the organic one, described in the book by Maxine Bedat, "Unraveled: The Life and Death of a Garment". It includes some historical notions that contribute to raising both scientific knowledge about the process of production of cotton and social awareness about working conditions, as well as environmental awareness about the effects of chemical products.

The students are, then, divided in groups of four, each one representing a farm in the year 1992 (the one before Carl changed cultivation mode). Each group is a team and has the goal of growing the farm they represent. Each farm starts with the same budget, with a conventional cultivation mode and with ten regular employees. At every round, which represents one year, the only two choices possible concern the cultivation mode (either conventional or organic) and the type of contract (either regular or undeclared). Following the example of Carl, the students in the game perform actions that address environmental (the former) and social (the latter) aspects of cotton cultivation. At every round of the game, each farm computes the revenue of the corresponding year. Every year, however, unexpected events occur and constitute an advantage/disadvantage for those farms who have a conventional/organic cultivation mode, and/or a regular/undeclared type of contract for their employees. The goal of the activity is to understand why there are a few farms that produce organic cotton in the world.

It is possible to outline two different uses of mathematics by the students in the game: in the lesson observed for this research, two groups out of six used mathematics as a tool for reasoning and long-term prediction (for example: “if a field today costs 16k, in a number of years can bring me a certain gain”, noting however an impossibility to recoup the costs within twenty years). These two teams were able to make choices informed by their computations, but they were slow. The remaining four groups were faster, because they made choices without using mathematics as a tool, focusing solely on the events of the present, but suffering the consequences of choices made without calculating in advance their effects on the long term. For these students, mathematics remains somehow hidden and unhelpful in predicting the effects of their strategies. For the other two groups, mentioned above, mathematics surfaces as a need, namely as a tool necessary to inform their strategies and win.

The authors also noticed that, in order to theorise and apply models of reality, the students should develop a deeper awareness of the world that surrounds them and a critical spirit, useful to improve their cognitive understanding. In this process, the use of mathematics is crucial: its upstream application, central for the construction of graphs from which students can draw information, but ‘hidden’ behind the graphs, helps them to create a clear picture of pollution, in the example showcased. A direct application of mathematics is instead essential to investigate the complexity of reality, as it resulted from the game in the context of cotton: who has performed more calculations has been able to model a more distant reality in time, those who were satisfied with the least effort have 'suffered' the consequences of their choices year by year. A goal is to experience a rigorous application of mathematics, which may or may not be central, in order to overcome the proposed oversimplified models from mass media, useful for a basic smattering, but insufficient to investigate a complex reality. A second goal is to inform the university mathematics curriculum with activities that bring socio-ecological issues to the forefront and, through simulation of real cases, provoke awareness and deep thinking about such issues.

Having now considered examples of curriculum innovation in community/school context and in undergraduate mathematics, we now move to the same question in relation to teacher education. As already noted, curriculum change, by definition, places new requirements on teachers. In advocating for socio-ecological practices of mathematics education, what this means for the curriculum of teacher education is a critical question to consider.

Teacher education innovations

To support mathematics teachers in the kinds of transformations required to engage in socio-ecological practices, mathematics teacher educators themselves need to consider what it means for mathematics teacher education curriculum to be shaped by the socio-ecological. Firstly, there is the question of what constitutes the *content* of teacher education curricula that teacher educators need to consider. Content considerations include which theoretical concepts to introduce, which classroom examples and curriculum innovations to utilise, which mathematical activities to engage teachers in, and which socio-ecological issues to address. For instance, introducing mathematics teachers to concepts from a critical mathematics education perspective could present rich opportunities for mathematics teachers to reflect on their current practices and to consider how they could develop a more critical stance in themselves and their mathematics students (e.g. Steffensen et al. 2021). Furthermore, engaging mathematics teachers in existing activities that link mathematics to, say, issues of social justice, could provide a shared experience for collaborative groups of teachers to build upon in developing curriculum innovations relating to issues of relevance for their own students (e.g. Wright, 2021). Barwell and Hauge (2021) establish a set of key pedagogical principles for "the development and organisation of mathematics classroom activities related to climate change" (p. 182) which could provide a basis for the curriculum content of mathematics teacher education and professional development. The authors are careful not to present these principles as a "definitive set of empirically validated standards" (p.172) and recognise the interpretation, application or modification of their key principles will vary from place to place. Their words imply the importance of mathematics teacher educators working alongside mathematics teachers to develop their own sets of principles, meaningful within their own local contexts.

For teachers to transform their practices, to orient themselves towards the socio-ecological, requires teacher educators to consider not only *what* content to engage mathematics teachers in and with but also *how* mathematics teachers can be supported in enacting new ways of being mathematics teachers. Thus, there is a need for research that focuses on innovative pedagogical approaches within teacher education and professional development, approaches that support teachers in re-envisioning their mathematics classrooms for sustainable futures. In this regard, there is an increasing recognition that the arts and humanities can provide opportunities to engage with wider socio-ecological issues and to alter attitudes and behaviours in ways that formal scientific approaches on their own do not. It has been argued that teaching and learning approaches involving a variety of art forms and aesthetic elements have qualities that could develop progressive teacher education (Møller-Skau & Lindstøl, 2022) especially in relation to a crisis like climate change, which is commonly perceived as both distant and abstract as well as "overwhelming and difficult to grasp" (Raven & Stripple, 2021, p.223). Research suggests that environmental, social and sustainable education relies on ways of reducing ambiguity whilst simultaneously maintaining uncertainty. To work in this paradoxical way is to embrace multiple contradictions and to draw on multiple forms of knowing (i.e. cognitive, sensible, somatic, affective) that arise through engaging within the materiality of aesthetic learning experiences. These experiences "invite the sensation of mind/brain/body simultaneously in both suspension and animation in the interval of change from the person one has been to the person that one has yet to become" (Ellsworth, 2005, p.17).

Kraehe and Brown (2011) explored how arts-based inquiries in social justice-oriented teacher education can provide generative spaces for developing prospective teachers' critical socio-cultural knowledge. Within mathematics teacher education, Helliwell and Ng (2022) utilised speculative fiction as a curriculum innovation, when working with prospective mathematics teachers from the UK and Hong Kong. Initially, the researchers had conceived of the use of speculative storytelling primarily as a way of engaging pairs of mathematics teachers (each pair consisting of one prospective teacher from the UK and one from Hong Kong) in sharing their ideas and collaborating around the topic of sustainable futures in the mathematics classroom. On further inspection, the mathematics teacher educator-researchers recognised the potential of speculative fiction as a pedagogical tool for prompting Skovsmose's (2015) notion of pedagogical imagination. Skovsmose explains that it is through pedagogical imagination that "one tries to conceptualise alternatives to what is taking place" (p.114), hence the link to transforming classroom practices:

> Through a pedagogical imagination one tries to conceptualise alternatives to what is taking place – for instance in terms of ways of organising: interactions in the classroom, the content of the curriculum, the tasks set for homework, etc. A pedagogical imagination can help to reveal that certain educational facts are not necessities but contingencies. I refer to this form of revelation as a modulation of facts. A modulation indicates spaces for possible changes, and I find that modulation forms a principal part of a critical activity (p.114).

According to Greene (1995), social imagination is "the capacity to invent visions of what should be and what might be […] in our schools" (p.5). It is this kind of imagination that enables a suspending and letting go of taken-for-granted ways of being to contemplate more just and equitable futures. Helliwell and Ng (2021) found that the stories created by the pairs of prospective mathematics teachers uncovered what the teachers considered to be changeable aspects of their own practice, i.e., what was contingent, and what they deemed as fixed. By revealing contingent practices, possibilities for new and different ways of being mathematics teachers were made available. The process of creating the stories was a process transforming the teacher's own self-envisioning.

It is well accepted that an interdisciplinary perspective is necessary to respond to complex problems such as climate change and associated social injustices (i.e. climate justice). It is currently a major challenge within existing frameworks (e.g. school curricula organised in discrete subject disciplines, and external assessments focussed on the mastery of subject specialised skills) for productive interdisciplinary work to take place in schools, especially at secondary level. There is thus a potential for teacher education institutions to take a leading role in this area by providing opportunities for prospective teachers from across different subject disciplines to collaborate, to learn from one another and to plan interdisciplinary projects for students to engage in (e.g. Hawkey et al. 2019). Helliwell et al. (under review) report on a small-scale professional development project for secondary mathematics teachers on teaching mathematics and climate justice which was designed and facilitated by an interdisciplinary team of researchers consisting of two mathematics teacher educators, a research mathematician, a practicing site responsive artist, and a novelist and creative writer. The professional development was based on a series of arts-based activities including utopian and dystopian writing, creating art-installations, and creative mapping exercises. The authors, who include one of the participating mathematics teachers, present a layered text as both the process and product of analysing the teacher's transformation as a mathematics teacher for climate justice in relation to the professional development activities (re-calling Nicol et al. 2022). In producing this layered text, the authors develop a framework they call teaching mathematics 4 climate justice (*TM*4*CJ*) consisting of four dimensions (teaching mathematics *about, with, for, as* climate justice). The project reveals the multiple ways in which teaching mathematics and climate justice can be conceptualised and the authors emphasise the importance of teacher educators and teachers working together to develop frameworks relevant within their own local contexts.

DISCUSSION

Looking across the descriptions of the three areas of curriculum innovation, we now consider some similarities and differences. One of the ideas that emerges from our combined research lies in the area of models and how these models are created and used. On the one hand, models can obscure and conceal, and working to illuminate contextual omissions is a part of the work described above. On the other hand, as cited above, Skovsmose (2021) states that models are performative; this idea of performativity implicates all of us as getting involved and being engaged. Models do not stand apart from people, they infiltrate our own thinking and ways of doing. And so much of the described research above is about performative engagement of mathematics and socio-ecological concerns which, in and of itself, is a model which includes inquiry, discussion, mathematising and a commitment to an on-going interrogation of collective experience.

In a plenary panel address for the ICMI Symposium on "Mathematics Education and the Socio-ecological", which took place on 20th March 2023, Mogens Niss asked two questions which are relevant to this Research Forum. He asked: (1) What can socio-ecological issues do for mathematics education? And, (2) What can mathematics education do for the dealing with socio-ecological issues? Niss' own answer to these questions focused on mathematical models – their creation, use and critique – as ways that socio-ecological issues can influence mathematics education, and for mathematics education to influence socio-ecological issues. This is clearly one area for further research work, in relation to socio-ecological questions. Work already done within socio-critical modelling (e.g., Araújo, 2010) offers important insights, in this regard.

Educational innovations can present experiences for exploring solutions to socio-ecological dilemmas and related pedagogical tensions. The research we have presented pushes for awareness of ecological and social challenges and constitutes possible solutions that support ownership and responsibility, relying on mathematics. Nevertheless, the work we have reviewed also points to several dilemmas and tensions, linked to several aspects of complex socio-ecological issues, such as the tensions between school and community interests and tensions for teachers engaging in such work. As we have presented, when working with Indigenous communities, teachers can be concerned about not teaching tasks aligned with Indigenous perspectives or appropriating Indigenous knowledge, even from a participatory research approach. Teachers may also experience difficulties or barriers in relating the mathematics curriculum to interdisciplinary projects and problems.

Taking socio-ecological issues to the mathematics classroom implies new relationships and responsibilities between mathematics, school and communities, which impacts curricular designs, teachers' education and classroom educational activities design. We believe research in mathematics education needs to continue studying such relationships and exploring solutions, theoretical and methodological perspectives, and experiences to face severe socio-ecological global challenges. In the Research Forum itself, we hope to raise such tensions and questions and engage in dialogue with participants.

CONCLUSION AND NEXT STEPS

For this Research Forum, we come together and share our thoughts towards how mathematics can integrate with socio-ecological issues and socio-ecological practices. As discussed in the introduction, we consider three aspects that we believe are necessary for furthering thinking about socio-ecological issues and mathematics: (i) increase awareness of ecological and social crises, (ii) raise ecological, socio-economic and socio-political dilemmas and related pedagogical tensions, (iii) explore solutions that rely on mathematics, and support mathematical development, ownership, and responsibility. Through the writing of this report, we have also begun to elaborate on practices of engaging in research which seem necessary for addressing socio-ecological questions in a manner that respects their complexity.

We suggest that attention to the socio-ecological can help us understand issues in a deep way, can guide us toward solutions and also can support learning mathematics (among other things). Therefore, we see socio-ecological issues as a necessary part of a mathematics curriculum. In the areas of research described above, we see a variety of approaches. One study describes students interrogating graphs and exploring what is factually conveyed so as to illuminate bias and misconceptions about water toxicity in Italy. Another study described students mathematising a socio-ecological disaster to convey the notion of the magnitude of the disaster, so that students have a personal connection with the dramatic event. Another study engaged community groups, scientists and educators, in creating materials of use and benefit to the school and wider community. We view these approaches as appropriate to the context and needs of both teachers and students. There will surely be no "one-size fits all" socio-ecological practice.

One of the ideas that emerges from the work above is that it extends mathematics, as a defined curricular discipline, to a lived experience. If one asks, "where is the mathematics?", one is already detaching mathematics from this lived experience. We have argued here in the research described above, that mathematics is less about how it might be defined in the curriculum and more about engagement with our world. If mathematics is about the transformation of a scenario to something more symbolic, formal, quantitative, qualitative, then engaging with the world seems to be the perfect forum. The research above describes how we might see and how we might practice in a way that does not put a check box curriculum as a priority, but, rather, provides an opportunity for students to engage with aspects of the world they are experiencing, in the now, as well as contemplating lived experiences of future generations.

We also note a theme which has recurred through the description of research, which is the significance of story and storying. The power of story emerges in methodological practices that are sensitive to Indigenous contexts and in the use of arts-based approaches to working with teachers. Stories offer mechanisms for "unlearning to relearn" (Sultana, 2022, p.122). Stories can allow a connection to the past, to the present and to possible futures. And here we see a strong link with Gutiérrez's (2021, 2022) call for a "re-storying" of mathematics, within a "spiritual turn" for our field. As part of this spiritual turn, Gutiérrez poses a series of questions, including: "(What are our underlying theories of change?) … What kinds of (mathematical) futures are we making? And what does that say about who we are becoming as researchers or persons?" (2022, p.382). This Research Forum has not touched on theories of change and we acknowledge the challenge of articulating such theories as an important issue within any socio-ecological practice of mathematics education. And invoking a sense of the future is also a thread through all the work reviewed above. It seems perhaps inevitable that addressing any socio-ecological question brings us up against questions of how the future might be otherwise.

Reviewing past work within mathematics education has indicated that few scholars have directly addressed socio-ecological questions, and yet, as we have tried to highlight, many related fields have developed practices which offer insights and starting points for addressing the complexity of socio-ecological issues. There is a question about how the socio-political turn in mathematics education relates to the socio-ecological ideas we have presented. The relationship appears rather complex. While a socio-political perspective seems in many ways to encompass socio-ecological practices, there is also something in the socio-ecological which may point to new questions, as we suggested in the introduction, such as consideration of a drained lake or a polluted river. One gesture we note within socio-ecological practices is a de-centering of human concerns (Coles et al. 2022), for instance, in placing a polluted river at the centre of a research project, at the centre of activism, and at the centre of a "memorial museum". One act, in which we might engage, is a "re-storying" of mathematics education research, in imagining how our work and our concerns could shift or alter, in bringing to the centre of attention socio-ecological concerns.

We do notice signs of change. The theme of PME46, for example, is an excellent case of an increasing focus on questions of sustainability within mathematics education. We hope this Research Forum will act as a further impetus to the field to pay attention to socio-ecological practices. We feel convinced that the pressures brought on by a warming planet will mean the curriculum has to change and one way we view the work of the Research Forum is that of preparation for a curriculum to come.

Our plan for the Research Forum sessions is as follows:

Session 1: Introduction & review of existing research foci, philosophical perspectives and frameworks (20 minutes + 5 minutes questions).

Activity relating to a project on Moana-nui-a-kiwa Indigenous communities' socio-ecological challenges. This section will have a methodological focus and participatory research methods will be offered actively (25 minutes).

Presentation of a project involving the co-construction of a curriculum with students, teachers, and families in marginalised communities in Mexico (20 minutes + 5 minutes questions).

Discussion of issued raised and on-going research questions (15 minutes).

Session 2: Review of session 1 and introduction to session 2 focus (5 minutes).

Presentation of work on an undergraduate sustainability course and an example of reasoning about social issues. Discussion on the role of mathematics in understanding environmental issues, reflections on the teacher's role (30 minutes + 10 minutes discussion).

Activity based on research that utilises arts-based approaches to working with teachers of mathematics. Speculative writing methods will be used as a way of supporting participants in reimaging their own work (25 minutes).

Final discussion of issues raised, implications and next steps (20 minutes).

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INNOVATIVE RESEARCH APPROACHES TO MATHEMATICS TEACHER NOTICING

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In recent years, teacher noticing has gained prominence as a theoretical construct in mathematics education, highlighting the dynamic, situational aspects of teaching that underlie instructional decisions and actions. This research forum explores innovative research approaches to teacher noticing in mathematics education, focusing on four key areas: theoretical perspectives and conceptualizations of teacher noticing, methodological approaches to the study of teacher noticing, teachers' professional learning of noticing, and new research directions in teacher noticing.

BACKGROUND AND AIMS

In the last two decades, teacher noticing has gained considerable prominence in the education literature (König et al., 2022), especially regarding mathematics education (Dindyal et al., 2021). One reason for this is that it underpins teachers' decision-making, which relies on teachers paying attention to and interpreting instructional details (e.g., students' mathematical thinking, including critical thinking; equity and inclusion) to make informed decisions about how to proceed in their lessons. It is therefore not surprising that teacher noticing is increasingly recognized as a fundamental aspect of teachers' professional competence (Kaiser & König, 2019).

The education field today is shaped by various perspectives, which have diverse implications for how noticing is conceptualized and thus studied (Scheiner, 2021). The study of noticing poses considerable methodological challenges, especially since noticing is difficult to capture and make explicit (Kersting et al., 2016). However, technological advances offer new approaches for accessing and assessing teacher noticing and for developing noticing expertise (Kosko et al., 2022; Weyers et al., 2023). Video has been used extensively to support teachers in developing their noticing skills (Santagata et al., 2021), but alternative and emerging approaches may offer new and different ways of supporting teachers in learning to notice (Amador et al., 2021; Walkoe et al., 2020). Noticing skills are influenced by teachers' knowledge, beliefs, and practices, and are socially and culturally shaped in important ways.

In this research forum, these issues are discussed in detail, focusing on innovative approaches and their potential for development. The research forum is organized around four key areas: (1) theoretical perspectives and conceptualizations of teacher noticing, (2) methodological approaches to the study of teacher noticing,

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(3) teacher professional learning of noticing, and (4) new research directions in the field. Each of these areas will be discussed by a group of senior and emerging scholars in mathematics education, followed by the discussants' commentary to further stimulate developments in the field.

ESTABLISHED AND EMERGING THEORETICAL PERSPECTIVES ON TEACHER NOTICING

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This paper offers an overview of the theoretical perspectives on and conceptualizations of teacher noticing—a critical and complex activity that is vital for effective teaching. We explore established and emerging cognitive–psychological, expertise-related, discipline-specific, sociocultural, and ecological-embodied perspectives, discussing each perspective in terms of how noticing is conceptualized within that perspective. The paper concludes by recommending a multiperspective approach that may better capture the intricate and nuanced nature of noticing.

INTRODUCTION

Teacher noticing is a complex phenomenon that has been explored from various theoretical perspectives, each of which emphasizes different aspects of this crucial activity for effective teaching. Some scholars regard teacher noticing as a set of interrelated mental processes, while others emphasize the differences between novice and expert teachers' processing of visual information in the classroom. Some researchers highlight discipline-specific practices or socioculturally organized forms of professional vision, with others viewing teacher noticing as an embodied form of exploration that involves interacting with the classroom environment. These diverse perspectives have prompted rich, ongoing discussion about the nature of teacher noticing, what it entails, and how it can be developed and studied. The present paper builds upon a recent literature review and uses a theoretical framework that incorporates an additional perspective on teacher noticing to examine how each perspective defines and conceptualizes this complex activity (König et al., 2022).

Although the following overview of theoretical perspectives is not all-encompassing, it endeavors to present a useful framework for understanding the distinctive characteristics of each perspective and how they enhance our understanding of teacher noticing. Through this exploration, we aim to elucidate the key similarities and differences among various approaches to teacher noticing.

A COGNITIVE–PSYCHOLOGICAL PERSPECTIVE

A cognitive–psychological perspective conceptualizes teacher noticing as a set of perceptual and cognitive processes that teachers use to observe and make sense of noticeable incidents in the classroom. Scholars who have adopted this perspective have identified various such processes, often drawing on van Es and Sherin's (2002) initial conceptualization of noticing, which includes identifying noteworthy classroom situations, connecting specific classroom interactions to broader teaching and learning principles, and using contextual knowledge to rationalize classroom interactions.

Research that takes this perspective has used various conceptualizations of noticing to distinguish between the different processes that underpin teacher noticing of classroom events, their interpretations of those events, or, in some cases, the decisions that arise from what is attended to and interpreted (Jacobs et al., 2010; Kaiser et al., 2015; Sánchez-Matamoros et al., 2019; Star & Strickland, 2008). Despite these variations, a shared theme among these studies is an interest in how teachers "construct" what they see (Erickson, 2011), linking observed events to abstract categories and characterizing their observations in terms of familiar instructional episodes (Sherin et al., 2011). Many of these studies have been based on information processing models, seeking to uncover the cognitive processes involved in teacher noticing, as well as the knowledge required for teachers to make sense of classroom events (Scheiner, 2016; Sherin & Star, 2011).

AN EXPERTISE-RELATED PERSPECTIVE

An expertise-related perspective on teacher noticing focuses on novice–expert differences in perceiving, processing, and monitoring visual information within the classroom. The roots of this perspective can be traced back to early research by Berliner (1988) and Carter et al. (1988) on differences in experts' and novices' perceptions and understanding of classroom information. These studies were precursors to the more recent discourse on teacher noticing, although they did not explicitly use the noticing construct. This research highlighted that expert teachers possess a more extensive repository of classroom knowledge than novice teachers and that their information processing differs, enabling them to evaluate significant classroom incidents more effectively and establish meaningful connections with their knowledge and practical experience, which in turn allows them to act more adaptively.

Building on this foundation, recent research has focused on the cognitive processes and resources involved in expert teachers' noticing and has identified different noticing profiles among teachers with varying levels of experience (Bastian et al., 2022; Jacobs et al., 2023). In addition, recent conceptualizations, such as the cognitive theory of visual expertise posited by Gegenfurtner et al. (2023), have been employed to model the visual information processing of experts.

A DISCIPLINE-SPECIFIC PERSPECTIVE

Noticing from a discipline-specific perspective involves intentionally directing attention and sensitized awareness toward particular aspects of one's teaching practice. This can be achieved by systematically evaluating one's own teaching practice, as described by Mason (2002), with the aim of increasing awareness of one's own actions, questioning habitual reactions in specific situations, and sensitizing oneself to future opportunities to act with intentionality rather than automatically out of habit.

Mason's (2002) construal of noticing outlines a set of practices for developing teachers' sensitivity and presence in the classroom based on having "a reason to act" and "a different act in mind" (p. 1). These practices include: (1) systematic reflection based on recording important moments and retrospectively identifying threads, (2) recognizing typical situations and formulating alternatives, (3) preparing and noticing by sensitizing oneself to possibilities for action and enhancing opportunities for noticing, and (4) seeking validation from others by describing moments and refining tasks to highlight important issues or sensitivities (Mason, 2002, p. 95).

The discipline of noticing involves the lived experiences of teachers and is phenomenological in nature (Mason, 2011, p. 231). These four practices bring "the moment of noticing from the retrospective into the spective, into the moment, so that a choice can be made to respond rather than to react habitually" (Mason, 2002, p. 87). By intentionally directing their attention and sensitized awareness, teachers become more methodical and intentional in their practices without becoming mechanical or reactive.

A SOCIOCULTURAL PERSPECTIVE

A sociocultural perspective on teacher noticing highlights that noticing is not solely a psychological process, but also a socially situated activity shaped by discursive practices and sociopolitical contexts. Goodwin (1994) argued that professionals, including teachers, develop "professional vision," meaning socially organized ways of seeing and understanding events that are specific to the teaching profession. Professional vision is shared and negotiated through historically constituted practices that enable professionals to construct "objects of knowledge" based on the phenomena that interest them.

Goodwin (1994) identified three practices involved in the formation and communication of professional vision: coding, highlighting, and producing material representations. Coding involves translating observed phenomena into relevant knowledge objects for a particular profession, highlighting makes specific phenomena more prominent by marking them in some way, while producing material representations involves creating external cognitive artifacts that organize and display relevant knowledge.

Professional vision is perspectival, positional, and ideological, with the authority to organize the field of vision unevenly distributed. Power relations are implicated in professional vision, as certain ways of seeing and understanding are privileged over others (Lefstein & Snell, 2011).

Recent studies on teacher noticing have highlighted the cultural, historical, and ideological contexts in which noticing occurs, as well as its relationship to broader discourses and systems (Dreher et al., 2021; Louie et al., 2021; Shah & Coles, 2020; van Es et al., 2022). However, Goodwin's original approach has been widely disregarded, or even, in some cases, lost (for a discussion, see Louie, 2018).

AN ECOLOGICAL–EMBODIED PERSPECTIVE

An ecological–embodied perspective on teacher noticing considers noticing to be an embodied activity of exploring and engaging with the classroom environment. This approach has been introduced quite recently, expanding upon more established perspectives on teacher noticing (Scheiner, 2021). According to this perspective, teachers gather information through continuous interaction with the material and social aspects of the classroom. This viewpoint is informed by Gibson's (1979) ecological approach to perception, which emphasizes the active role of the perceiver in perception and the fundamental reciprocity between a perceiver and the environment.

Gibson (1979) understood perception as a dynamic process because it involves active exploration of the world; it is "an experiencing of things rather than a having of experiences" (p. 239). Gibson (1979) argued that individuals modify their environment and alter its affordances to better suit their needs or intentions. Similarly, teachers shape the classroom environment to make specific events noticeable, establish particular interpretations, or bring certain events into being.

The ecological–embodied perspective on teacher noticing highlights the interdependence of perceiver and environment, the reciprocity of perception and action, and a form of direct perception. It posits that teachers are active participants in the instructional scene, not just passive observers.

Recent studies on teacher noticing have highlighted the embodied and ecological nature of noticing, providing insights into how teachers gather information by moving their eyes, heads, and bodies to interact with the classroom environment (Jazby, 2021; Kosko et al., 2021; Scheiner, 2021). For example, Scheiner (2021) characterized teacher noticing as "an embodied way of accessing, exploring, and engaging with the world of classroom events" (p. 88), emphasizing its dynamic and immersive nature.

CONCLUSION

Research on teacher noticing has drawn on various theoretical perspectives, including cognitive–psychological, expertise-related, discipline-specific, sociocultural, and ecological–embodied perspectives. Although the cognitive–psychological perspective is the most prevalent in the literature (König et al., 2022), it is crucial to appreciate the contributions of each perspective toward a more comprehensive understanding of how and why teachers notice what they do.

Each of the five perspectives offers a particular definition of noticing. The cognitive–psychological perspective views noticing as a set of mental processes that individual teachers engage in when observing and making sense of classroom events. The expertise-related perspective explores noticing in terms of changes in visual information processing that lead to more adaptive classroom behavior as teachers gain expertise. The discipline-specific perspective sees noticing as deliberate and systematic attention to one's own teaching practices aimed at increasing sensitivity to future opportunities for intentional rather than automatic action. The sociocultural perspective regards noticing as a socially situated activity shaped by discursive practices and sociopolitical contexts. Finally, the ecological–embodied perspective considers noticing to be an active, embodied process of exploring and engaging with the classroom environment, emphasizing the reciprocity of perception and action.

It is important to recognize that these different perspectives are not mutually exclusive. Instead, they offer complementary lenses through which to examine the complex phenomenon of teacher noticing. For instance, the sociocultural perspective expands the cognitive–psychological perspective by acknowledging the sociocultural influences that shape individual psychological processes. Future research may investigate how the development of individual teacher noticing shapes, and is shaped by, the evolution of professional noticing among the community of practitioners to which they belong. This would deepen our understanding of the sociocultural influences that shape the psychological aspects of noticing, and vice versa.

Similarly, the ecological–embodied perspective can enrich cognitive–psychological and sociocultural perspectives. An ecological approach concentrates on how teachers encounter and act in their environments, revealing the automaticity with which they notice in meaningful ways. This approach necessitates examining not only what is inside teachers' minds but also what their minds are inside of. Noticing is not strictly internal or external but combines aspects of both, reflecting an ecological commitment.

As research on teacher noticing continues to evolve, researchers may study this phenomenon from various theoretical perspectives to develop a comprehensive understanding of the phenomenon. Adopting multiperspective approaches may allow researchers to posit multidimensional explanations of noticing that account for its sensual, positional, relational, and political nature.

METHODOLOGICAL APPROACHES TO THE STUDY OF TEACHER NOTICING

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In this paper, we discuss different methodological approaches to the study of teacher noticing, review established approaches focused on written and spoken noticing, and examine knowledge-based constructs. Additionally, we present new methodological advances, including the use of 360-degree videos and other extended reality technologies. We discuss the challenges and innovations for each approach, emphasizing the need to connect novel methods to established theory for a cohesive understanding of teacher noticing.

OVERVIEW

Noticing is a complex and multifaceted construct that involves differentiating between the incoming visual-auditory input from the classroom environment that is attended to and that which is not. Efforts to examine this complex phenomenon, as well as the factors associated with it, have been predominately qualitative, but quantitative and mixed-methods approaches have become more prevalent in recent years (König et al., 2022). Methodological approaches include the form of data collected and the theoretical aims of examining it. Herein, we discuss different methodological approaches to the study of teacher noticing, focusing on established approaches to examining written and spoken noticing as well as more recent approaches, such as those for examining knowledge-based constructs and incorporating extended reality technologies.

METHODS FOR EXAMINING WRITTEN AND SPOKEN NOTICING

The written or spoken expressions of teacher noticing are commonly examined in an attempt to deal with the complexity the internal and implicit processes underpinning it. Researchers have used diverse data collection and analysis methods in line with their theoretical perspectives and preferred interventions, each with its own merits and limitations in achieving the researchers' goals (Nickerson et al., 2017; Santagata et al., 2021). A common approach to investigating noticing is to provide teachers with a particular artifact (e.g., a written classroom episode, a lesson video, etc.) as a specific context to notice. Using a shared artifact allows for comparisons between teachers. Usually, the activity is accompanied by guiding prompts, either open/general or specific, to which the teachers are asked to respond in writing or orally (individually or as a group) according to the given prompts. Generally, open-ended questions are used to obtain rich responses, but closed online questionnaires facilitate larger-scale data collection. Another method invites teachers to reflect, either in writing or through recorded interviews, on events they have noticed during their own teaching. Although such an approach makes comparison difficult, it more closely simulates teacher noticing in the classroom.

Data collection may be conducted at a single point in time, usually with the aim of exploring teachers' observations in a certain setting. Alternatively, it can be conducted at two or even more different time points to identify changes (or developments) in teacher noticing, such as those fostered by participation in a professional development course.

Along with diverse data collection methods, there are diverse methods of data analysis. Some researchers "openly" search for what the teachers notice in an event. For example, van Es and Sherin (2008) examined what teachers noticed in video segments of lessons in terms of the participants and the content (e.g., mathematics, pedagogy, etc.). Rotem and Ayalon (2022) extended the existing lenses to capture additional (emotional and social) dimensions of an instructional event, as expressed in teacher noticing. Other researchers sought to examine teachers' noticing of a particular aspect of teaching and learning, such as students' mathematical thinking (Jacobs et al., 2010), or a specific mathematical practice, such as argumentation (Ayalon & Hershkowitz, 2017). The analyses in such cases are sometimes based on certain criteria, according to what the researchers would like the teachers to notice in the event. Along with *what* the teachers notice in the event, researchers often also assess *how* they notice, usually using a rubric to help them evaluate the quality of the written or spoken expressions of noticing, such as the extent to which it is evidence based and the level of specificity.

Investigating teachers' written and spoken noticing can be challenging. One of the challenges is that we can only form an approximation of noticing as reflected in the teacher's writing and/or oral communication. Another challenge relates to the generalizability of noticing measurements in a given context. Yet another challenge lies in measuring what the teachers *do not* notice, in addition to what they do notice. Researchers are trying to cope with these challenges by developing innovative methods that will allow them to better understand teacher noticing skills and design interventions for developing these skills.

METHODS FOR EXAMINING KNOWLEDGE-BASED ASPECTS OF NOTICING

Teachers rely on several resources to facilitate their professional noticing, one of which is teacher knowledge, and approaches that examine the knowledge-based aspects of teacher noticing are becoming more prevalent (König et al., 2022). Some scholars have examined the interactions between professional knowledge and teacher noticing based on quantitative approaches (Jong et al., 2021; Yang et al., 2021), but others have incorporated qualitative (Dick, 2017; Kooloos et al., 2021) and various additional approaches that can best be described as "novel" (Cross-Francis et al., 2022; Kersting et al., 2021).

Quantitative approaches typically incorporate formal measures of mathematical knowledge for teaching and employ quantitative indicators of noticing. For example, Yang et al. (2021) used open- and closed-response items to assess 203 Chinese teachers' content and pedagogical content knowledge (PCK), and they quantitatively scored teachers' written descriptions based on video vignettes to conduct correlational and path analyses. A similar approach was incorporated by others (Jong et al., 2021; König et al., 2014) using data collected through open and closed questions to assess teachers' professional knowledge and specific coding schemes to evaluate their written responses to brief video clips. Qualitative approaches to examining knowledge-based aspects of noticing allow for more detailed analyses of how professional knowledge interacts with and facilitates components of teacher noticing. A common approach involves examining teachers' discourse patterns as they attend to and interpret students' mathematical thinking from either written work or video (Dick, 2007; Kooloos et al., 2022). These approaches provide detailed analyses of how teachers' knowledge is operationalized to notice students' thinking at the expense of smaller details.

Over the past decade, various scholars have attempted to move beyond these established methods to capture additional aspects of teacher noticing. For example, Kersting et al. (2021) examined moment-to-moment noticing, which they "conceptualized as a noticing task that was filtered through teachers' knowledge of their own practice" (p. 110). Analysis of this moment-by-moment noticing of video clips indicated more nuanced aspects than did holistic ratings by teachers of the videos alone. Cross-Francis et al. (2022) examined teachers' stimulated recall based on their own recorded teaching and found that their emotions about and beliefs toward mathematics significantly affected how they operationalized their professional knowledge in the noticing of their own instruction. A third approach was developed by Jacobs and Kruschke (2011), who used Bayesian networks to examine coded elements of teacher knowledge and noticing extracted from written and verbal responses.

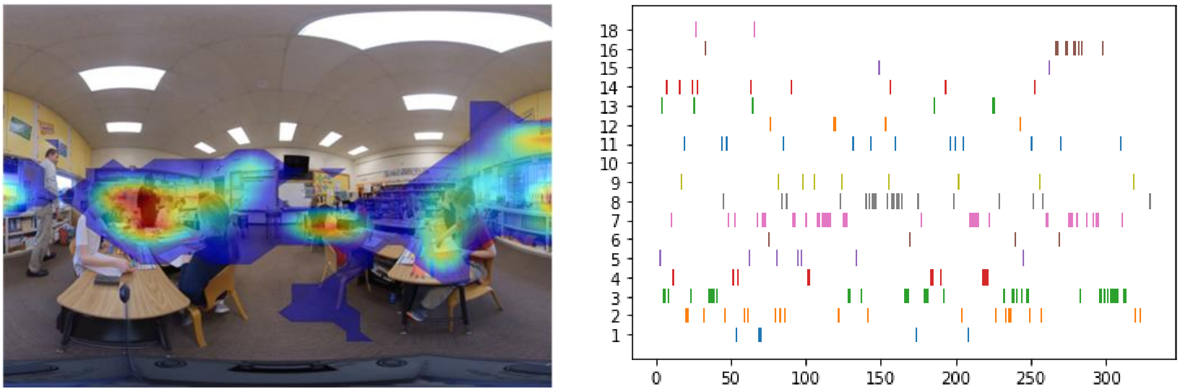
METHODS FOR EXAMINING EMBODIED ASPECTS OF NOTICING

Teacher noticing has been described as teachers making sense of the "blooming, buzzing confusion of sensory data" (Sherin & Star, 2011, p. 69) in the classroom. Teachers perceive much of this sensory information, but noticing involves "selecting stimuli perceived in a scene (Scheiner, 2016, p. 231) and disregarding other aspects perceived in a pedagogical environment (van Es & Sherin, 2021). Because established methodologies for analyzing teacher noticing do not examine physiological factors, many scholars have started incorporating novel technologies to study embodied aspects of teacher noticing, including measurements of physiological data, such as eye and head movement tracking (Huang et al., 2021), and creating immersive contexts for exploring teachers' enacted noticing through virtual reality and 360 video (Kosko et al., 2021; Weston & Amador, 2021).

Methodologies incorporating eye and head tracking make use of the visuospatial system described by Gibson (1966). Most research using eye tracking to study noticing has focused on differences between preservice and in-service teachers, observing that experienced teachers' gaze durations are shorter than those of novice teachers (Huang et al., 2021), but they look at more students in the classroom and for similar amounts of time (van den Bogert et al., 2014). For example, Huang et al. (2021) found that preservice teachers tended to focus on the students closest to them, and in-service teachers looked at students who were both proximally closer and further away. Extending these findings, Kosko et al. (2023) examined total gaze time and found that preservice teachers with higher assessed PCK gazed for longer at students farther away, while those with lower PCK gazed for shorter periods.

Similar to eye-tracking approaches, scholars have used 360 videos to examine where teachers look (Kosko et al., 2021; Weston & Amador, 2021). Because 360-degree video provides a panoramic recording of the classroom, the viewer must move the camera perspective to focus on certain areas of the classroom. This technological affordance has been used to collect both qualitative and quantitative data on where teachers look. For example, Weston and Amador (2021) had preservice teachers record their mathematics teaching and watch 360 videos with the researcher across multiple teaching sessions. One teacher was initially self-focused, but then turned and looked at what students were doing during the lesson. Similarly, other scholars have examined teachers' fields of view and found that an increased focus on students rather than the classroom teacher coincided with more descriptions of what the mathematics students were learning (Kosko et al., 2022). Such trends can also be examined quantitatively. For example, Gandolfi et al. (2022) prepared count data for the regions of the classroom on which teachers focused in a 360 video. The count data were used to measure unalikeability—a nonparametric indicator of variance for nominal data—which was then employed in a regression model for statistical analysis. Kosko et al. (2023) used a combination of eye-tracking and field-of-view data to record teacher noticing in a 360-degree video (see Figure 1). The total gaze time per student group was then correlated with the participants' PCK scores.

Figure 1. Eye-tracking data from a 360 video presented by Kosko et al. (2023)



The various approaches described here for examining embodied aspects of noticing focus predominantly on the visuospatial system and include both qualitative and quantitative aspects. However, other physiological data can be and has been examined, including auditory (Ferdig et al., in press) and cardiovascular (Huang et al., 2022) data. Common to these approaches is the aim of examining not only the embodied factors involved in noticing, but also how they correspond with traditionally analyzed data on noticing (i.e., written/spoken data and knowledge-based factors).

CONCLUSION

Our review of methods for examining teacher noticing provides a snapshot of both established and emerging approaches. Some approaches, such as those examining knowledge-based aspects of noticing, have been emerging for some time but continue to attract innovation. Others, such as those focused on the embodied aspects of noticing, are more nascent. In each case, these emerging methods are often used in conjunction with or extended from established approaches to examining written and spoken noticing. Such connections are advantageous for scholars seeking to combine novel and established methods. However, a further advantage is theoretical since each of these approaches relies on different resources that teachers may draw on in the act of noticing (cognitive, embodied, etc.). As noted by Scheiner (2021), these resources are connected, and scholarship that highlights these connections in both theoretical and methodological frameworks may enable researchers to gain a more cohesive understanding of the overarching nature of teacher noticing.

EXPLORING REPRESENTATIONS AND THEIR AFFORDANCES FOR SUPPORTING TEACHERS' LEARNING TO NOTICE

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This paper explores different representations of practice and tools and their affordances in supporting teachers' learning to notice. The paper is divided into three main sections, focusing on vignettes, simulations, and video annotation tools. We discuss the affordances and constraints of these representations and tools and how they can be effectively leveraged to support teachers' learning experiences.

INTRODUCTION

Teacher noticing has been the focus of much work in mathematics education. This previous research has considered noticing processes (attending, interpreting, and deciding how to respond), as well as how these processes can be developed in teacher education programs.

A variety of representations of practice have been used to support the development of teacher noticing and have taken different forms, such as cartoons, videos, depictions, and written case studies (Chazan et al., 2018; Fernández et al., 2018; Ivars et al., 2020). Professional development (PD) experiences that employ these various representations have been investigated, designed, and deployed to support teacher noticing. To this end, tools (including annotation software) have been combined with representations to enhance teacher learning. For instance, video tagging and annotation tools have allowed teacher educators and researchers to explore the development of teacher noticing over time (Walkoe & Levin, 2018; Walkoe et al., 2020).

In the following, we discuss ways of using various representations and tools to support teacher noticing. In particular, affordances (and limitations) of various representations and platforms and how they can be leveraged to support teacher learning experiences are addressed.

VIGNETTES TO SUPPORT PRESERVICE TEACHERS' NOTICING OF STUDENTS' MATHEMATICAL THINKING

Noticing students' mathematical thinking involves knowledge-based reasoning processes, since preservice teachers must use their knowledge (about mathematics and the teaching and learning of mathematics) to attend to, interpret, and decide. In previous work, we explored the characteristics of learning environments in teacher training programs that can help preservice teachers develop this competence (Fernández et al., 2018; Ivars et al., 2020). The results showed that vignettes of learning environments are potential tools for developing preservice teachers' noticing of students' mathematical thinking. A vignette provides a representation of practice, some questions to guide the analysis of the representation, and an explanation of the theoretical knowledge required to interpret the representation (Ivars et al., 2020).

Representations of practice provide preservice teachers with real contexts and opportunities to relate theoretical ideas to the practice of mathematics teaching. They are understood as depictions of classroom situations (e.g., depicting different students' responses to an activity or illustrating teacher–student interactions when engaged in activities). Guiding questions (prompts) are used to focus preservice teachers' attention on noticing the specific mathematical details of students' answers (selective attention), interpreting students' mathematical thinking based on the identified details, and deciding on a learning objective and activity to help students develop their understanding. Theoretical research on mathematics education related to students' mathematical thinking (acting as a theoretical lens; Fernández & Choy, 2020) provides the required knowledge to attend to, interpret, and decide. In some learning environments, this information follows a learning trajectory. The use of these theoretical lenses is justified since previous research has shown that preservice teachers can learn to attend to important details of students' thinking but may have difficulties in using these details to interpret students' thinking or make specific decisions (Fernández & Choy, 2020). Therefore, the development of teacher noticing in teacher education programs is challenging without a guide to direct teachers' attention to important aspects of the students' answers, provide them with the required knowledge to interpret student's mathematical thinking (such as different levels of students' mathematical understanding, students' difficulties, etc.), and provide them with information to define specific learning goals and activities to help students advance their understanding of, for example, task variables, levels of difficulties in the activities, and the importance of different activities.

Previous research considered the development of preservice primary and secondary school teachers' noticing of students' thinking in different mathematical domains using learning environment vignettes. In this paper, the characteristics of such learning environments and vignettes that can support preservice teachers' development of noticing students' mathematical thinking are discussed.

REPRESENTATIONS OF PRACTICE: AFFORDANCES AND CONSTRAINTS

For at least the past two decades, video-based technologies have been the predominant technologies used to represent teaching practices and to support teacher noticing in mathematics (Santagata et al., 2021). More recently, researchers have begun using other types of multimedia technologies to represent practices (e.g., animations or lesson sketches) and develop noticing competence (Herbst et al., 2011). However, few researchers have focused specifically on the affordances of different representations of practice for supporting teacher noticing. Affordances refer to specific functions that can be activated based on the properties of a representation (Hatch & Grossman, 2008). The few studies that have examined the affordances of representations have largely focused on video-based representations (Dindyal et al., 2021), and far fewer have focused on the nature of representations in relation to teacher noticing outcomes (Superfine & Bragelman, 2018).

The effectiveness of learning from representations of teaching depends on the affordances of the representations themselves and the settings in which they are analyzed. Representations of practice vary in the granularity of the teaching and learning interactions they make visible. Moreover, learners (e.g., teachers) not only need to analyze teaching and learning in ways that are appropriate for their experience levels but also need to observe what is present—or not present—in the representations (e.g., details that might be obscured; Hatch & Grossman, 2008). Moreover, representations of practice should capture the realities and complexities of teaching and learning interactions. How can accessible and authentic representations of practice be provided to learners?

In our research, we explored the role of different representations of practice in supporting preservice teacher noticing. We borrowed the concept of the "decomposition of practice" from Grossman et al. (2009) and applied it to our work on noticing to decompose noticing practices into their constituent components. To decompose practices and enable novices to "see," how can we leverage the affordances of different representations of practice to support preservice teachers' noticing of students' mathematical thinking in accessible and authentic ways? No single representation can accomplish such goals. Instead, we need a repertoire of different representations of practice with different affordances that can be used to decompose the practice of noticing. Different representations make students' thinking more or less visible; in other words, they vary in opacity. In our research, we focused on three main categories or types of representations of practice: static representations that capture the "end products" of students' thinking (e.g., student work samples), dynamic representations that illustrate students' thinking processes (e.g., video clips or audio recordings of problem-solving think-aloud sessions), and malleable representations that allow for intentional interactions with a "student" to probe and influence the elicitation of students' thinking in real time (e.g., for teaching simulations, see Shaughnessy & Boerst, 2018). In this paper, we discuss the unique affordances of each category of representations for supporting preservice teacher noticing.

USING VIDEO ANNOTATION TOOLS TO SUPPORT NOTICING IN VIDEO CLUBS

Video clubs are effective professional development interventions to support noticing (Sherin & van Es, 2009). In prior video club work, teachers watched a short classroom video clip together, and a facilitator then led a discussion about the students' thinking they noticed in the video (Sherin & Han, 2004; van Es & Sherin, 2008). A transcript was provided to support the teachers' analysis of the video, and they were allowed to take notes while viewing that they could refer to during the subsequent discussion. In the discussion, the participants primarily relied on their memory of the video and any notes they took as they discussed student's observed thinking.

A few issues arose from this approach. First, as the teachers watched the video, their attention was split between the video and the transcript. When teachers looked at the transcript, they often missed important nonverbal clues to students' thinking in the video. Second, since the participants watched the video synchronously, they could not pause the video or rewind it without the facilitator's intervention, and interesting nonverbal cues could easily be overlooked or forgotten. Given the significance of the nonverbal aspects of students' thinking in cognition (Goldin-Meadow, 2003), we consider it important to provide more support for multimodal teacher noticing in video club PD (Walkoe et al., 2023).

In a current project, in which we are using a video annotation platform (www.anotemos.com) (Herbst et al., 2019) to support multimodal teacher noticing (Walkoe et al., 2023). The platform allows teachers to view and annotate an embedded video clip prior to a video club discussion. We are investigating two types of annotation: (1) a "pin" feature that allows users to mark and comment on a moment of interest, indicated by a pin icon in the video progress bar located at the bottom of the video, and (2) a "draw" feature that allows users to pause the video and draw on the screen. Such annotations are saved in the video and visible for a few seconds before and after the drawing is completed. Both the "pin" and "draw" features allow users to add comments.

Some affordances ask teachers to watch and annotate a clip, unlike previous methods whereby teachers watched a video together and it was only paused or rewound at someone's request. The Anotemos tool gives teachers the ability to watch a video on their computers and pause or replay the video if desired. Another major benefit of watching and annotating video clips is that teachers' full attention is on the clips rather than split between the video and a transcript. We found that teachers are able to pay more attention to nonverbal aspects of students' thinking and discuss them more deeply in video clubs when they use the video annotation tool. This is important because our work indicates that when teachers attend to nonverbal aspects of students' thinking, they also rationalize the students' thinking in more substantive ways using a more resource-based lens (Walkoe et al., 2023).

In addition to allowing teachers to pay attention to nonverbal cues, we have also found ways to use the annotation tool itself to prompt teachers to attend to nonverbal aspects of students' thinking. We found that when teachers used the draw tool to highlight moments they noticed, they only attended to students' nonverbal cues, which contrasted with their behavior when they used "pins" to tag examples of students' thinking when they only attended to verbally expressed ideas. We continue to explore this promising support for video clubs.

DISCUSSION

Central to all three sections is a focus on the role of different tools and representations of practice in supporting teachers' noticing of students' mathematical thinking. Each tool or representation has particular affordances for supporting teacher noticing. For example, a video annotation tool affords opportunities to attend to both students' verbal mathematical thinking and nonverbal mathematical thinking (e.g., gestures and drawings). Using the annotation tool, teachers can interpret students' thinking in substantive ways grounded in evidence. Vignettes, on the other hand, leverage focal questions and information about students' thinking depicted in representations to support teachers in attending to and interpreting student thinking, but also afford opportunities to decide subsequent learning objectives and activities for students based on the mathematical thinking represented. Moreover, representations of practice, such as simulations, afford opportunities for teachers to interpret students' mathematical thinking and to elicit and potentially influence students' thinking by virtue of the questions they pose. Indeed, different tools and representations, and the contexts in which they are used, can be leveraged to support the development of effective noticing processes.

Additionally, the current work raises several issues regarding the use of different tools and representations to support teacher noticing. Although many tools and representations of practice have unique affordances for supporting teacher noticing, they also have some limitations. The affordances of particular tools or representations may not be aligned with the needs of novice learners (i.e., preservice teachers) who have limited knowledge about teaching and learning interactions. Moreover, the use of different tools and representations requires facilitators/teacher educators to leverage the unique affordances of each tool or representation.

NEW DIRECTIONS IN RESEARCH ON TEACHER NOTICING

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In this paper, we explore new directions in research on teacher noticing, focusing on three aspects: inclusive mathematics education, the relationship between beliefs and teacher noticing, and sociocultural influences on teacher noticing. We highlight the importance of teacher noticing in facilitating inclusive education and student-centered teaching. We also emphasize the relationship between teachers' beliefs and their noticing skills, showing that constructivist beliefs about teaching and learning mathematics positively impact teachers' noticing abilities. Furthermore, we present findings from cross-cultural comparative studies, demonstrating that cultural characteristics influence what teachers notice and how they interpret what they do and make decisions in the classroom.

INTRODUCTION

Over the past decade, teacher noticing has attracted considerable attention from researchers around the world. Recent studies have focused on understanding the underlying principles of and new directions for noticing research, including empirical studies in the field of inclusive education. Furthermore, efforts have been made to investigate the relationship between teacher noticing and teacher beliefs, or the dimensionality of cognitive and situated aspects of teachers' professional competence. Moreover, researchers discuss the cultural characteristics of teacher noticing (Louie, 2018) and investigate its strengths and weaknesses by comparing teacher noticing in different contexts (Yang et al., 2021). In this paper, we explore new directions in research on teacher noticing, focusing on three aspects: inclusive mathematics education, the relationship between beliefs and teacher noticing, and sociocultural influences on teacher noticing.

TEACHER NOTICING IN INCLUSIVE MATHEMATICS EDUCATION

In recent years, inclusive education has been identified as a key challenge for education by policymakers and organizations worldwide (OECD, 2019). It is considered to meet the needs of all students and to take a broader perspective on dealing with heterogeneity. Heterogeneous classes, diverse learners in diverse situations, and the wide variety of subject matter pose specific challenges for teachers, particularly for their diagnostic or formative assessment competencies, their intervention skills, and, thus, their teacher noticing (König et al., 2017; Vantieghem et al., 2020).

Teacher noticing can be understood as an important skill for facilitating inclusive education and student-centered teaching. Teacher noticing is an important component of teachers' professional skill sets and is increasingly included in conceptualizations of teachers' professional competencies for connecting knowledge and other factors, such as beliefs, with teachers' behavior in the classroom. Recent results have emphasized the crucial role of teacher noticing skills in students' learning success and identified it as a characteristic of expert teachers (Blömeke et al., 2022; Jacobs et al., 2010; Lachner et al., 2016). Therefore, understanding and fostering teacher noticing are key for teacher education and professional development, particularly for inclusive education.

Nonetheless, only a few studies have investigated teacher noticing in the context of equity and diversity (König et al., 2022), and even fewer have specifically addressed inclusive (mathematics) education or ways of dealing with heterogeneity. However, there are some exceptions. Keppens et al. (2019) and Roose et al. (2018) developed two video-based instruments—for primary and secondary education, respectively—using comparative judgments and rating scales on general pedagogy to assess preservice and in-service teachers' perceptions and interpretations of positive teacher–student interactions and differentiated instruction as two main aspects of inclusive education. The results revealed connections between beliefs and teacher noticing skills in inclusive education (Keppens et al., 2021) and highlighted some influences at the school level on noticing, stressing the relevance of schoolwide, internal efforts to incorporate inclusive teaching practices (Roose et al., 2019). Moreover, high proficiency in teacher noticing in inclusive education seemed to correspond with the implementation of communication-promoting and differentiating teaching practices (Gheyssens et al., 2019). Beyond these studies, Meadows and Caniglia (2018) analyzed noticing skills in the co-teaching practices of inclusion- and content-specialized teachers. Their results showed that teachers' beliefs about teaching, learning, and collaboration during co-teaching were aligned, whereas the noticing focus (i.e., the topics that were noticed) differed and were connected to the teaching profession (Meadows & Caniglia, 2018). Besides studies specifically addressing inclusive education, some works have focused on equity, investigating noticing in the context of sociopolitical diversity dimensions, such as culture, ethnicity, or socioeconomic background (Louie, 2018; Shah & Coles, 2020), as a way of examining inclusive education practices.

However, to our knowledge, no studies have investigated how teachers' noticing skills can be developed to support inclusive education or what knowledge might be necessary to facilitate this. In addition, domain-specific perspectives appear to be missing from the discussion. These desiderata are addressed by the *Teacher Education and Development Study – Inclusive Mathematics Education* (TEDS-IME) project, the participants of which have created a professional development program for inclusive mathematics education that is currently being implemented. Accompanying pre–post design evaluations using newly developed video-based teacher noticing instruments based on existing instruments (Kaiser et al., 2015) and knowledge tests for inclusive education have provided promising insights concerning the mentioned research desiderata.

RELATIONSHIP BETWEEN BELIEFS AND TEACHER NOTICING

There is great interest in noticing research on the features that influence teacher noticing ability and development, such as sociocultural aspects, cognitive dimensions, and affective and motivational components. Teachers' beliefs are a crucial facet of the affective–emotional domain (Blömeke & Kaiser, 2017). Studies incorporating situation-specific skills (i.e., perception, interpretation, and decision-making) into their analyses have found connections between these skills and teachers' beliefs for different teacher groups and in various settings; for example, preschool teachers (Dunekacke et al., 2015), primary school teachers (Larrain & Kaiser, 2022), secondary school teachers (Griful-Freixenet et al., 2020), and inclusive settings (Keppens et al., 2021; Roose et al., 2019).

In a large-scale study in Germany involving 131 primary school teachers (a follow-up study to the International Teacher Education and Development Study; TEDS Follow-Up), researchers analyzed the interrelations between teacher noticing skills, their subject-specific knowledge, and beliefs about the teaching and learning of mathematics. They assessed teacher noticing (perception, interpretation, and decision-making skills) using three short video clips from primary school mathematics teaching and corresponding questions. In addition, they assessed teachers' mathematics and pedagogical content knowledge, as well as their constructivist beliefs. Analyses revealed that teacher noticing was significantly correlated with their beliefs and subject-specific knowledge, indicating that teachers holding constructivist beliefs about the teaching and learning of mathematics perceived opportunities that provided insights into students' thinking more precisely and analyzed them more competently (Hoth et al., 2022). A multi-regression analysis showed that these beliefs had a greater influence on teacher noticing than their professional subject-specific knowledge (Hoth et al., 2022). They also showed, again, the great influence of beliefs on the teaching and learning of mathematics and teacher noticing in particular.

How exactly this interconnection between teachers' beliefs and noticing ability occurs and what role beliefs play in the development of teachers' professional competence are crucial issues for the education field and should be the focus of further research. Such research may provide deeper insights into how teachers' professional competence is formed and how it influences their performance in the classroom.

SOCIOCULTURAL INFLUENCES ON TEACHER NOTICING

As mentioned previously, teacher noticing has been accepted as a socially and culturally shaped construct (Louie, 2018). Therefore, mathematics teachers working in different sociocultural contexts may notice differently in practice. In recent years, in a few cross-cultural comparative studies, researchers have compared the similarities and differences in mathematics teacher noticing between the West and the East. For example, Yang et al. (2019) carried out a comparative study in China and Germany that involved a large sample of in-service mathematics teachers. The findings of this study showed that compared with in-service German mathematics teachers, in-service Chinese teachers performed significantly worse on general pedagogy-related aspects but much more strongly on mathematics instruction-related aspects. Further differential item functioning (DIF) analysis results revealed that German teachers demonstrated specific strengths in aspects of the perception process, such as classroom management and the behaviors of students and teachers. In contrast, Chinese teachers demonstrated specific strengths in aspects of interpretation and decision-making processes, such as using knowledge to analyze and reason about incidents. For noticing in mathematics instruction, DIF results showed that German in-service mathematics teachers were better at noticing aspects such as mathematical modeling and visual approaches to teaching. In contrast, Chinese in-service mathematics teachers demonstrated specific strengths in using knowledge to make judgments about students' work, evaluate students' mathematics mistakes, and develop alternative lesson plans.

Similarly, Ding et al. (2023) compared the similarities and differences in teacher noticing between expert elementary school mathematics teachers in China and the United States. They found that at the macro level, the expert mathematics teachers from both countries paid great attention to the teaching domain, such as by using representations, deep questions, and classroom communications. However, the researchers identified differences at the micro level; for example, American mathematics teachers noticed more about concrete representations, whereas Chinese expert teachers noticed more about the sequence of representations (e.g., from concrete to abstract). In addition, the comments made by American teachers contained less reasoning than those made by Chinese teachers.

In summary, the findings of cross-cultural comparative studies on mathematics teacher noticing have provided empirical evidence that teachers in different sociocultural contexts tend to perceive different aspects of mathematics teaching and interpret their perceptions differently; correspondingly, they also make different decisions. Generally speaking, Chinese mathematics in-service teachers tend to pay more attention to teaching content and demonstrate specific strengths in knowledge-based reasoning. In contrast, teachers in Western countries, such as Germany and the United States, tend to perceive more general pedagogy-related issues (Yang et al., 2019).

CONCLUSIONS AND RECOMMENDATIONS

To understand teacher noticing and decode its development and impact as a crucial part of teachers' professional competencies, various aspects should be considered. Not only are inter- and intrapersonal aspects important, but also the context and environment in which teacher noticing takes place. Studies have suggested that teacher noticing is shaped by individual beliefs and shared contextual conditions. Teachers' constructivist beliefs about mathematics teaching and learning have a positive effect on their noticing skills. In addition, cultural characteristics crucially influence what teachers notice, their interpretations of classroom situations, and the decisions they make. Moreover, additional but critical contextual aspects, such as heterogeneous settings, seem to shape teacher noticing. Nevertheless, how these different elements interact and how each influences the development of teacher noticing remains unclear.

Therefore, future research on teacher noticing should consider both its intrapersonal and contextual aspects in relation to theoretical frameworks and empirical methods. Interactions between intrapersonal factors, such as knowledge, skills, and beliefs, and between intrapersonal aspects and contextual factors should be further studied to enrich our understanding of teacher noticing in the classroom. In addition, the characteristics of teacher noticing in different areas of teaching and their dependence on these areas need to be considered. Studying similarities and differences in teacher noticing in inclusive settings or in different areas of mathematics teaching and learning would provide further insights into the construct, its development, and its impact on students' learning.

TEACHER NOTICING: A PASSING FAD OR HERE TO STAY?

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It is clear from the contributions to this research forum that there have been many innovative research approaches to studying teacher noticing in various contexts. As the authors of the various contributions have claimed, research on teacher noticing may be at a critical point in its development trajectory. Will such research continue on an upward trajectory, or will it dwindle and come to the end of its innovation cycle? In this commentary, I argue that research on teacher noticing may continue to grow if our research continues to strengthen the different "powers" of teacher noticing as a foundational construct to explain, model, and develop different aspects of mathematics teaching across various contexts.

MANY INNOVATIONS, BUT …

There have been many innovative research approaches to studying teacher noticing in various contexts, as highlighted by all the authors participating in this research forum. Important aspects of this development are the different theoretical perspectives that underpin studies of teacher noticing. As Scheiner and Kaiser (included herein) explained, noticing has been conceptualized from at least five perspectives—cognitive–psychological, expertise-related, discipline-specific, sociocultural, and ecological-embodied. They recommend using a multiperspective approach to provide a more comprehensive view of teacher noticing, because each of these perspectives has afforded different nuanced explanations and models of teacher noticing.

The proliferation of these different perspectives has been complemented by a corresponding development in the methods used to examine teacher noticing. To this end, Ayalon et al. (included herein) reviewed both established and emerging methods and highlighted some recent innovations, such as eye-tracking technology and 360 videos, that can be used to capture the complexities of what teachers notice. More importantly, they argued that these innovations not only contribute to our repository of methodological tools but also to a more connected and cohesive view of the different theoretical perspectives on noticing. Correspondingly, the same repository of methodological tools provides mathematics educators with the means to develop tools and representations to support teachers' learning to notice. As presented by Fernandez et al. (included herein), representations of practice, such as animations and lesson sketches, together with annotations of video-based tools, have unique affordances for supporting teachers' development of effective noticing competencies. It is also crucial to note, as pointed out by Bastian et al. (included herein), that teacher noticing is influenced by contextual factors (e.g., sociocultural influences) and intrapersonal factors (e.g., beliefs about teaching and learning in a variety of contexts). Therefore, they argued that it is time for researchers studying teacher noticing to investigate the interactions between these influences and factors.

Despite these innovations in the study of teacher noticing, several significant concerns have been raised. For example, Fernandez et al. (included herein) questioned the particular affordances of different tools and representations in relation to the different needs of novice teachers and the role of facilitators/teacher educators in harnessing these affordances. Bastian et al. (included herein) also highlighted that how contextual factors and intrapersonal factors interact to influence the development of teacher noticing remains unclear. Furthermore, although the different perspectives on noticing may complement and deepen our understanding of teacher noticing, these perspectives have different epistemological foundations. These issues suggest that the trajectory of research on teacher noticing may have reached a critical point. Innovations as a proxy for growth may not always result in sustained and impactful research. Furthermore, novelty alone is a weak basis for sustained research. These issues beg the following question: Will research on teacher noticing continue its upward trajectory, or will it dwindle as a passing fad and come to the end of its innovation cycle?

TEACHER NOTICING: A FOUNDATIONAL CONSTRUCT?

With the aim of addressing these issues, it is crucial for us, as a research community, to carefully consider why teacher noticing is *foundationally* important in mathematics education—"to what extent will research on noticing help us better understand teacher practice and/or daily issues of teachers in the classroom?" (Dindyal et al., 2021, p. 11). To this end, it is useful to consider the explanatory, modeling, and generative powers of teacher noticing as a construct for understanding teaching. This follows from one of the key motivations behind the study of teacher noticing: “teachers’ changing practices are accompanied by new and enhanced teacher noticing” (Sherin et al., 2011, p. 11). In other words, teacher noticing is consequential. Hence, critically, the construct of teacher noticing can help explain what teachers can or cannot do in their classrooms. Conversely, closely associated with its explanatory power is its power to model a teacher’s decisions and actions in different teaching contexts according to what has been noticed—or not (Choy, 2016).

It is clear that teacher noticing must continue to be a generative construct for the purpose of understanding mathematics teaching. There are at least two generative aspects to consider. First, research in teacher noticing should continue to generate relevant characterizations of noticing to provide impactful insights into the different aspects of mathematics teaching, such as task design (Choy, 2016) and argumentation (Ayalon & Hershkowitz, 2017), among others. Second, research on teacher noticing should focus on generating new tools and methods to access, assess, and support the development of this component of teaching expertise. Moreover, there is a need to expand the terrain of teacher noticing research in terms of its ability to account for differences in culture, contexts, and practices (Dindyal et al., 2021).

Is teacher noticing a foundational construct for understanding mathematics teaching? I think the answer is potentially affirmative when we examine state-of-the-art teacher noticing research as shared in this research forum. A possible hindrance, however, could be the lack of clarity in the conceptualization of noticing. Although Sherin et al. (2011) believed that research on teacher noticing “is too young to benefit from a single definition” (p. 10), they acknowledged that it is important for researchers to clarify their conceptualizations of teacher noticing for the field to move forward. Echoing similar views, Dindyal et al. (2021) highlighted that it is useful for researchers to make their conceptualizations of noticing clear to fully realize the explanatory and analytical power of noticing as a construct. Scheiner and Kaiser’s (included herein) explication of the five theoretical perspectives on noticing is certainly a positive step toward unpacking the “black box” of noticing (Scheiner, 2016, p. 231) and clarifying the conceptualizations of noticing for use in future studies.

Enhanced clarity, facilitated by explication of theoretical perspectives, would also support researchers in explicitly connecting their conceptualizations of teacher noticing with the methodological commitments made in their studies, which is necessary for furthering “our understanding of the affordances and constraints that are linked to various conceptualizations of noticing” (Sherin et al., 2011, p. 10). As Ayalon et al. (included herein) claimed, there is a greater awareness among researchers of the need for explicit alignment between the methodological approaches used and theoretical stances on noticing. This alignment is key to ensuring that our findings harness the full explanatory and analytical powers of teacher noticing as a construct for understanding mathematics teaching. More importantly, current perspectives on noticing, especially the embodied perspective on noticing, have generated novel approaches to the collection of eye-tracking and head-movement data (Huang et al., 2021; Stahnke & Blömeke, 2021), and researchers are increasingly using virtual reality and 360 video to explore teachers' enacted noticing (Kosko et al., 2021). In addition to video-based methods, these advances in methodologies will certainly contribute to more varied and robust approaches to developing teacher noticing expertise throughout teachers' teaching careers.

This brings us to an important question concerning the relationship between the conceptualization of noticing and the generative aspects of noticing as a foundational construct: How do new tools and ways of developing teacher noticing expertise relate to our understanding of noticing as a construct? The different tools and representations described by Fernandez et al. (included herein) for decomposing teaching practices suggest a strong leaning toward cognitive–psychological and expertise-related conceptions of noticing. This is not surprising because these conceptions offer relatively straightforward explanations and models of how expertise in noticing may be developed to train teachers with different backgrounds in different contexts.

MOVING FORWARD: ISSUES AND POSSIBLE DIRECTIONS?

In summary, teacher noticing can be seen as a foundational construct to explain and model mathematics teaching, potentially generating new tools, representations, and approaches to capturing and assessing expertise in teaching. However, whether the trajectory of teacher noticing research can continue its upward trend depends on our collective efforts to build on and expand mathematics teacher noticing research. The move toward studying teacher noticing in other academic, social, and cultural contexts is another positive step. As highlighted by Bastian et al. (included herein), there have been some encouraging shifts toward understanding noticing in the contexts of inclusive mathematics education and different cultural backgrounds. There is a need to continue pursuing the study of noticing across more varied contexts and to be open to the possibility that expertise in noticing can be perceived differently from currently dominant thinking. Much more work is required. For instance, the links between knowledge, beliefs, contexts, physiological data (e.g., gaze patterns), and teachers' noticing expertise are rather tenuous. For research on teacher noticing to strengthen and grow, we need to be able to give an *account of* and *account for* (Mason, 2002) the interactions between teachers' knowledge, beliefs, and practices in relation to what they notice. This may also require extending the examination of noticing beyond in-the-moment noticing to include the whole gamut of teaching activities (e.g., lesson planning and reflection) and corresponding representations of practice, focusing on pre-teaching and post-teaching noticing (Choy, 2016). This research forum offers an exciting opportunity for us to critique and co-construct our understanding of teacher noticing as a foundational construct so that its explanatory, modeling, and generative powers can be fully harnessed in future research.

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PROBLEM SOLVING WITH TECHNOLOGY: MULTIPLE PERSPECTIVES ON MATHEMATICAL CONJECTURING

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Research on technology and mathematics education has been a longstanding interest of the PME community. In this paper we revisit the interplay between technology and conjecturing within the process of problem-solving with an intention to capture different aspects of the processes in which students make and explore mathematical conjectures, and roles that both technology and teachers can play in this process. The focus is two-fold: first, to discuss different interpretations of conjectures and conjecturing within mathematics education, as reflected selected current works in mathematics education research; and second, to offer a discussion on progress in the implementation of these ideas with considerations of developments in technology, and our wider understandings of the role of the teachers.

INTRODUCTION

> "I have dared to undertake a dangerous journey on the basis of a slight supposition and already see the foothills of new lands. Those who have the courage to pursue the exploration, will step onto those lands…" **Immanuel Kant** (1755, I:222)

> "All positive knowledge must be reached if at all by an operation that begins with conjecture" **Charles S. Peirce** (1910, p. 283)

Prologue

> "… the intellectual progress of mankind in mathematical and scientific domains depends on our being able to make and explore conjectures…how might we change the way we educate people so as to help them make and explore conjectures…?" (Schwartz, 1992, p. 167).

These 30-year old thoughts by Judah Schwartz still hold as a relevant issue in mathematics education nowadays. In the following, we revisit what conjectures are and We propose why conjectures can become powerful springboards for 'intellectual progress' (i.e. learning both mathematical content and learning to do mathematics). We also suggest and exemplify some implications to support students in engaging in the practice of making productive conjectures.

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Conjectures - What?

- "A mathematical conjecture is a proposition about a previously unsuspected relationship thought to hold among mathematical objects." (Schwartz, 1997, p. 95) [1]
- "Conjectures are ideas formed by a person (the learner) in experience which satisfy the following properties: the idea is conscious (though not necessarily explicitly stated), uncertain and the conjecturer is concerned about its validity." (Norton, 2000, p. 290) [2]
- "Inference formed without proof or sufficient evidence. A conclusion deduced by surmise or guesswork. A proposition (as in mathematics) before it has been proved or disproved." (Merriam-Webster Dictionary) [3]
- "(Verb) Infer, predict, form (an opinion or notion) upon probabilities or slight evidence" (Online Etymology Dictionary) [4]
- "A statement strictly connected with an argumentation and a set of conceptions where the statement is potentially true because some conceptions allow the construction of an argumentation that justifies it." (Pedemonte, 2001, p. 34) [5]

This eclectic selection of definitions sheds light on the multifarious nature of both conjectures and conjecturing. If we choose the one word that is central in any definition, we find at least two characterizations: a proposition, a statement (grammar), and an idea, an inference, a prediction (semantics). What is such proposition/idea/prediction about? A "previously unsuspected relationship" [1], "a conclusion" [3], a connection to "a set of conceptions". How does a conjecture arise? From the learners "experience" [2], deduced by "guesswork" [3], based on "probabilities or slight evidence" and generated by "some conceptions" [5]. What about the conjecturers? They are "conscious" of the idea even if not explicitly, "uncertain and concerned" about the validity of the conjecture, and possibly able to undertake "the construction of an argumentation that justifies it" [5].

Conjectures - Why?

The mottos (by Kant and Pierce) heading this paper provide a first general justification for engaging in conjecturing to support learning: it may lead us to explore "a new land", generating new knowledge along the way.

More specifically, conjecturing nudges the conjecturers (a) to harness perception, previous knowledge and explicit reasoning in order to be clearer about how they envision the situation they are working on; (b) to become the 'owners' of a conjecture, and thus more (cognitively and affectively) committed to it, and (c) to become expectant and motivated for acting in pursue of evidence to prove or disprove the conjecture (Arcavi, 2000).

In the pursuit to confirm or discard a conjecture, it may happen that unexpected or counter-intuitive results emerge creating a clear disparity between outcomes and explicitly stated predictions. Students working on such activities are described, for example, in Hadas and Hershkowitz (1999). Such a disparity can be the trigger for nurturing the students' own need for re-inspection of their knowledge and assumptions, even without the teacher prompting to do so, creating propitious opportunities for meaningful learning.

Conjectures - How?

Conjecturing, as many other habits of mind (in the sense of Cuoco, Goldenberg & Mark, 1996), requires curriculum materials with activities that lend themselves well to stimulate students to conjecture. Moreover, it requires the creation of a secure environment and enactment of appropriate classroom norms that legitimize and support conjecturing while containing students' negative affective reactions, such as refusal, frustration, lack of confidence and fear of incorrect predictions, which may expose their lack of knowledge. In the implications of his extensive research on students conjecturing, Norton (2004, p. 367) mentions "interaction with peers" and "agreeable dispositions" as the "positive factors in generating constructive conjectural activity". Moreover, he states that if "teachers encourage students to verbalize their conjectures … other students might build from those assertions. But it is equally important for teachers to encourage students to be skeptical about such assertions, attempting to explain why they are viable or not (ibid.)." However, for that to happen, the conjecturer should feel that it is safe to take risks and to have the courage to persevere (Kant), because conjecturing may be necessary to attain "positive knowledge" but it may not be sufficient (Pierce).

Beyond the creation of appropriate learning materials and instituting a suitable classroom culture, students need not only learn to conjecture upon the request of the tasks or the teacher, but they should also take the agency to generate conjectures on their own. Autonomously asking what-if or what-if-not questions, experimenting with variations and invariants, observing patterns, attempting to generalize and searching for justifications place conjecturing as a promising companion to problem posing, as in Judah Schwartz's vision.

Technological environments provide a most proper arena for such a vision, since it allows students to experiment by themselves, and "to appreciate the ease of getting many examples . . . , to look for extreme cases, negative examples and non stereotypic evidence . . ." (Yerushalmy, 1993, p. 82). Such environments can provide immediate feedback as "dry" consequences of the student actions which, in some cases, may be more effective than teacher reactions, not only because of the affective implications (lack of value judgment), but also because it may engage the motivation to restate a conjecture, revise it, experiment again and even induce a need for proof.

Research on technology and mathematics education has been a longstanding interest of the PME community (e.g. Ferrara et al, 2006; Laborde et al, 2006;

Sinclair & Yerushalmy, 2016). As mathematics educators, a common goal has been to use technology to support inquiry-based practices in mathematics classrooms. In this paper, we focus on the process of mathematical conjecturing as part of technology-enhanced problem-solving activity. Our focus on the conjecturing process provides a lens through which to chart the evolution of technology designs and implementations across different mathematical content domains and phases of education.

We rely on a set of quotes underlining a set of sub themes around which we structure this paper. These quotes (Schwartz, 1992) serve as a connecting thread for the ideas that will be presented. In addition, we extend the focus to include more explicit attention to the role of the teacher, a dimension that was not foregrounded in Schwartz' original writing. Specifically, we use the following four themes: The role of mathematical conjecture in the construction of knowledge; The relations between Problem-Solving and Problem-Posing; Reflecting on the consequences of users' actions; and The impact of Digital environments' on learners' mathematical knowledge and the role of the teacher. For each of the subthemes, we will introduce a main idea, and then expand and exemplify it through recent technology and research, discussing the progress and development of these ideas over the years.

THE ROLE OF MATHEMATICAL CONJECTURE IN THE CONSTRUCTION OF KNOWLEDGE

On the role of conjecture in constructing knowledge, Schwartz states that "We need to have succeeding generations ask naturally and spontaneously about everything they see in the world around them, "What is this a case of ?"" (Schwartz, 1992). One well known natural human activity is gameplay, which can be employed to create forms of backward reasoning, which allows students to produce conjectures and reasoning in a more 'natural' way thanks to technology.

In different research studies (Soldano et al., 2019; Arzarello & Soldano, 2019; Albano et al., 2021) it has been shown how the so-called Logic of Inquiry (Hintikka, 1999; LI in what follows) can facilitate the structuring of different types of games, which allow students to produce conjectures and reasoning in a more 'natural' way thanks to technology. In fact, such games can foster a process of mathematical concepts learning, which is cognitively resonant with students' attitudes, slowing down the difficulties with and misuse of terms or logic in conjectures since early exploration stages of inquiry (Luz & Yerushalmy, 2023), but at the same time is epistemologically sound from the mathematical standpoint.

The above 'naturality' promoted by LI has deep cognitive and epistemological roots, respectively due to the rich productions of abductions by students in such activities and to the status of the mathematical truths within the game-theoretical approach. On the one hand, they allow to properly explain how the gap between the conjecturing and the proving processes (Arzarello & Soldano, 2019, p. 222) is reduced because of the processes triggered by the games proposed through a smart use of technology and of careful interventions of the teacher; on the other hand, these results suggest promising ways of deepening further the research and the design of teaching/learning situations with digital tools.

Consequently, in this section we first base on previous researches to give a short picture of the LI landscape for approaching proof in the classroom through a problem-solving approach, then we sketch a possible development of the research, which can be the starting point for a discussion in this RF.

LI has been introduced from a typical logic context (Hintikka & Sandu, 1997) into mathematics education in order to face and possibly overcome the seemingly unbridgeable 'basic double gap' between students' argumentative and proving processes underlined by most of the literature about the teaching of proof (Boero et al., 1996; Selden & Selden 2003; Pedemonte, 2007; Boero et al., 2010; Stylianides et al. 2017). The gap has both epistemological and cognitive features: it has been so described by Soldano & Arzarello (2016, p. 10):

On the one hand, there is an epistemological gap between what is empirically perceived as true and what is logically valid within a suitable theoretical framework […]. On the other hand, there is a cognitive gap between a first arguing phase in students' productions (when they are asked to explore a situation and make conjectures) and the proving phase (when they are asked to prove their conjectures in a more formal way).

LI allows to face this double gap through the design of mathematical games played by the students, who must develop strategic rules in order to win. Specifically, students, in couple, have to interpret mathematical statements as debates between two players: one assumes the role of falsifier F, who tries through her/his actions to disprove it, and the other member of the couple assumes the role of verifier V, who tries to prove it. If we consider, for example, a sentence like "$\forall x \exists y P(x, y)$" the dialectic between F and V will develop in this way: F starts by showing to V a particular individual a, chosen in the most unfavorable situation. If V finds an individual b, such that P(a,b) is true, V wins that hand, otherwise F wins it. In this way, the process to describe the truth of a sentence becomes a dialectic process: each action hides a questioning-answering dynamic, ruled both by definitory rules (the rules of inference) and by strategic rules (the rules of well reasoning in the game). According to Hintikka (1999), any kind of activity directed to the reach of an aim can be conceptualized as a game between two players, and this is true in particular for mathematical theorems. Such games, when treating geometric arguments, can be transposed into DGS environments, appropriately exploiting the robust and soft constructions.

For a simple but emblematic example (Figure 1), one builds a generic quadrilateral ABCD and only the middle points F of AC, and E of BD and the intersection point G between AC and BD are built in a robust way (Figure 1). The Verifier's aim is to make E, F G, to coincide, while the Falsifier has to make it impossible. The Verifiers moves C and the Falsifier moves D. After some hands, for the students it is clear that the Verifier always wins and that this result occurs because (s)he can always move D so to make E, F, G to coincide; in such a way the Verifier always constructs a parallelogram (DC parallel to AB and AD parallel to BC). At this point, suitable questions of the teacher can guide students to argue that ∀ D ∃ C (G = E = F) and why it is so; afterwards, the teacher can guide students to elaborate arguments and possibly proofs that “in a quadrilateral ABCD the diagonal AC and BD bisect each other if and only if the quadrilateral is a parallelogram”.

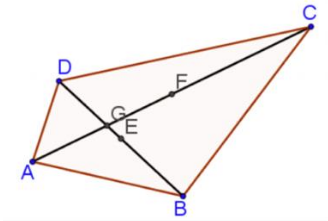

Figure 1: Initial state of the quadrilateral presented in the game

The truth of the sentence that students ascertain is based on a ‘backward approach’ rooted in the impossibility of building counterexamples to it: this strategy has an important epistemological and cognitive value.

The former has been discussed by Hintikka (Hintikka & Kulas, 1983), who elaborated a logical definition of semantic games, in which he reversed the standard definition of truth, given by Tarski (1933) and used in all textbooks of logic. In fact, Tarski’s definition starts from the condition of truth of the simplest (atomic) sentences and proceeds recursively to the complex ones: for example, to say if A&B is true one refers to the truth of A and B. The definition in LI is in the opposite direction: it starts with complex sentences and goes inside them, according to a top-down procedure, which is in accordance with the approach sketched out in the example.

The latter is discussed in Arzarello & Soldano (2019) and one of its main points (not the only one) consists in observing a phenomenon in students’ actions, productions and communications, which can generate a reduction of the above basic gap. In fact, semantic games extend the example spaces of students (Antonini, 2006) through their introduction of non-prototypical examples. Such non-prototypical examples are produced by students, typically by Falsifiers, with the aim to create difficulties to the Verifier. In the long run they produce reasoning to discover whether or not the Verifier can always win, since, to ascertain that, both players must check whether the geometric properties are still preserved. Consequently, the discussion of the new entries in their example space moves the attention from the figural to the conceptual aspects of the geometric figures (Fischbein, 1993), and activate their critical thinking (Abrami et al. 2015; Toulmin et al. 1984). We can therefore observe empirical evidence for a reduction of the basic gap in these discussions. The gap is also reduced since in semantic games two types of rules are used by students in a ‘natural’ way: the definitory rules, which are the possible and acceptable moves according to the game (deductive moves), and the strategic rules, which correspond to questions and answers for investigating which moves are the most convenient for a player in a specific situation (argumentative moves). There is so a deep interaction between the logic of justification and that of proving: the games are the cognitive and epistemic pivot, which allows this link between these two forms of reasoning to be made palpable. The mathematical theorems, transformed into games, embody a form of cognitive and epistemic continuity between the usual deductive logic of justification and the more argumentative logic of inquiry. This continuity constitutes a real added value to the educational games from the point of view of mathematics education.

We have based on LI for designing also a different type of games aimed at fostering students' learning of mathematical concepts in more formal contexts, e.g. in algebra, in a way which is cognitively resonant with their attitudes and epistemologically sound from the mathematical standpoint. We have called such games digital inquiry games: also in this case technology is crucial for their implementation. Space does not allow to enter into details and we defer to Albano et al. (2021) for their description.

As hinted above, the interplay between the logic of justification and that of inquiry is often characterized by the production of abductions (Peirce, 1878, p. 472 = CP 2.623; Magnani, 2001, 2009, 2015, 2023). The term refers to forms of reasoning that explain, and also sometimes discover some (possibly new) phenomenon or observation. Abduction is the process of reasoning in which explanatory hypotheses are generated and justified: in this latter sense, abduction is also often called 'Inference to the Best Explanation' (Douven, 2021).

Sometimes abductions regulate the actions made by the players in order to win and are produced within forms of backward reasoning (Gómez Chacón 1992; Shachter and Heckerman 1987; Beaney, 2021): in game-theory this way of reasoning is also called backward induction (it corresponds to what in chess is called retrograde analysis). This method is implemented, for example, in automated theorem provers, and its logical features were put forward by Hintikka (1998) and constitute the logical core of LI.

A careful analysis of abductions produced in technological games with respect to the used tools can suggest fuel for further research and for a discussion in RF, as we now shortly argument.

Magnani (2015) distinguishes between theoretical (or sentential) and manipulative abductions. Roughly speaking, theoretical abductions can be characterized at a logical level as those situations in which a hypothesis is formed and evaluated relying to the sentential aspects of natural or artificial languages. Theoretical abductions can be rendered in the Peircean syllogistic framework as transformation of a syllogism in 'barbara' (From a Rule, like 'all the A are B', and a Case, like 'x is an A', to a Result, like 'x is a B') into a fallacious syllogism (abduction), in which from knowing-selecting the Rule, and observing the Result, the Case is inferred. Such kinds of abductions, common in scientific reasonings, can be found in the students who play our games. However, many times, their abductions show also something more than a purely sentential production, which corresponds to forms of manipulative abductions. According to Magnani (2009), manipulative abductions are processes in which a hypothesis is formed and evaluated resorting to basically extra-theoretical behaviors, for example, manipulating diagrams in geometric reasonings. In our case, this happens within technological tools: the game creates a kind of an ‘epistemic negotiation’ between the internal framework of the student and the external reality of the diagrams built with the digital tool because of the proposed game. As claimed by Magnani (2009, p. 46), who in this relates to some of Peirce's observations (CP, 5.221): “manipulative abduction happens when we are thinking through doing and not only, in a pragmatic sense, about doing”. This is an exact picture of what happens in our games: students’ actions, e.g. when in the play with a parallelogram produce non-prototypical figures, have also an epistemic and not a merely performative role, which is relevant for abductive reasoning.

THE RELATIONS BETWEEN PROBLEM-SOLVING AND PROBLEM-POSING

The intellectual progress in the domains of mathematics and science depends on mathematicians and scientists “being able to make and explore conjectures. i.e. problems that we pose for ourselves” (Schwartz, 1992). In this section we demonstrate the relations between Problem-Solving and Problem-Posing through problem posing of various stakeholders in the learning process (students and teachers), and suggest possible perspectives connecting them to creativity and aesthetics.

Sinclair (2001) reports on the aesthetic dimension of problem posing, showing how students engaged in spontaneous problem posing as they explored and interacted with digital, colour-based representations of the decimal expansion of fractions. In early work, Sinclair (2001) explored the way in which an interactive, visual digital environment called *The Colour Calculator* (CC) could provoke grade 8 students’ problem posing. The work drew on the role that aesthetics plays in the problem posing of mathematicians (see Sinclair, 2004), especially in terms of how mathematicians drawn to certain patterns or objects as they experiment with mathematical objects. For example, the images in Figure 2 show different patterns produced by the fraction 1/7, shown in different table widths.

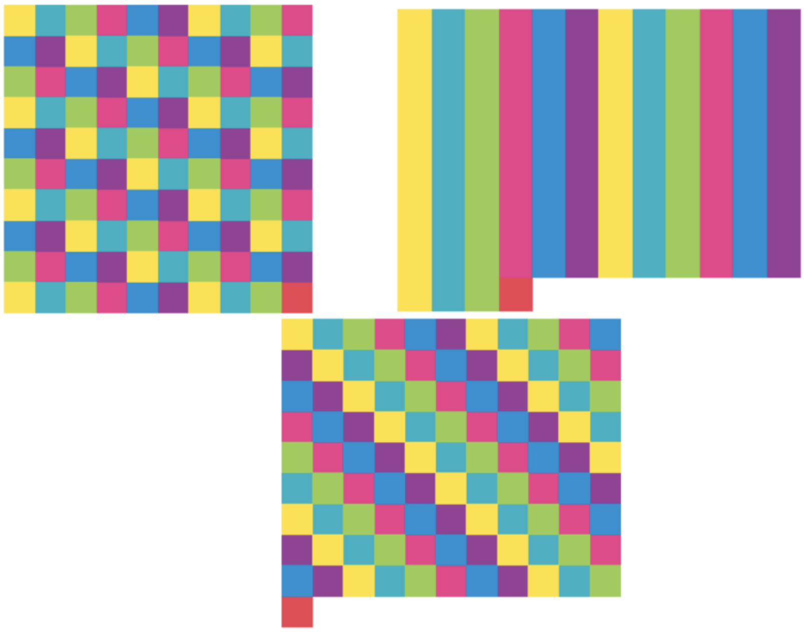

Figure 2: Patterns produced for the one seventh fraction

The students did indeed pose many problems that were mathematically interesting and that were provoked by the colour patterns and interactive options available in CC, such as: How to make a fraction that produces a table that is entirely red? I wonder how to make the diagonals go in the other direction? Is there a fraction that will not have a pattern? This finding, which corroborated Papert's (1980) view on the potential for non-experts to have aesthetic responses while engaging with mathematics. The aesthetic was operationalized less in terms of objective criteria (such as simplicity or generalizability or elegance) and more in terms of the visual appeal, thereby mobilising aesthetics more in relation to sensory knowing, such as the sensibility to visual patterns and colours.

Similar work on the aesthetic aspect of children's problem posing can be found in Eberle (2014) and Jasien et al. (2022), though in contexts involving symmetry and tessellations using non-digital technologies. One significant affordance of the digital environment of CC is the fact that it provides both symbolic and visual display and feedback, thereby enabling students to pose problems in the register of visual patterns yet connect them to numerical (in this case, fractions) registers. In their extension of the CC into sounds, which was developed for blind learners, Fernandes et al. (2011) report on similar aesthetic engagement, this time focused more on sound patterns and notes.

More recently, Sinclair and Ferrara (2021) draw on Whitehead's notion of prehension, which are subconscious aesthetic feelings, to study the way in which aesthetics is involved in student problem solving when they use *TouchCounts* (TC), a multitouch app for number sense. This work broadens the sensory scope to include the haptic/gestural experience of interacting with TC, as students use their fingers and gestures to create and manipulate cardinal quantities, which are then shown visually, aurally and symbolically. As with CC, it is therefore possible for students to draw either on the visual, oral, symbolic or haptic aspects of their mathematical interactions.

These multiple sensory presentations can be seen as way to aestheticise mathematics—by which I mean, make it more accessible to the senses—and, in so doing, to potentially draw on different ways of sensing and making sense. Other studies that were not specifically focused on problem posing did reveal some of the ways in which aestheticization provoked problem posing. For example, in Smythe et al. (2017), kindergarten children pose problems such as: "How can we make 100?", after having made 33, "What other numbers have the same digits?", "Are there different ways of making 10?", and "Can I make a number that's bigger than the screen?"

Significantly, in both CC and TC (as well as the environment studied by Eberle and Jasien et al.), there are no fixed tasks or given problem—the pedagogical goal of these environments is to engage students in explorations and experimentation, through which they might notice patterns or surprises that lead to the formulation of questions. While learners can generate such questions, it is another matter to determine whether the questions are themselves mathematically interesting, or even of aesthetic interest. For example, the question mentioned above, "Can I make a number that's bigger than the screen?" does not have much mathematical interest in the sense that it doesn't lead to any patterns or generalisations. Part of the process of mathematics enculturation could involve enabling students to become aware of what counts as mathematically interesting, something which teachers sometimes do in indirect ways (see Sinclair, 2008), and which Lehrer et al. (2013) have explored in more explicit ways.

Related to this, Crespo and Sinclair (2008) conducted research with elementary pre-service teachers in order to explore their ideas of what makes a mathematical problem interesting, as well as to inquire into how to promote their own problem posing. With respect to the latter, they found that having time to explore (with a set of tangram pieces) enabled pre-service teachers to pose more interesting problems than if they had to pose them without exploration, which confirmed Hawkins' (2000) suggestion that it takes time for learners to get to know an environment enough in order to be able to notice what might be interesting (unusual, surprising, compelling) to them. Here "more interesting" related to more focused on reasoning than on facts.

Tymoczko (1993) argued that mathematics needed to engage more in explicit aesthetic criticism in order to draw attention to certain features of a work of mathematics (such as a proof or a technique or a problem), which may enhance our appreciation of that work. Sinclair (2022) suggests that the same could be true in the context of mathematics education, where engaging students in questions of what is interesting or appealing might both enable students to appreciate the values that tend to dominate school mathematics (efficiency, unity, symbolism) and perhaps open up mathematical activity to other relevant cultural or individual values.

Turning to teachers, Problem posing-through-investigations (PPI), for example, is another mathematical activity in which investigations and conjecturing in a dynamic geometry environment (DGE) lead to posing problems and solving the posed problems (Leikin 2014; Leikin & Elgrably, 2020, 2022; Elgrably & Leikin, 2021). PPI tasks start from a proof problem (either introduced by instructors or researchers, or chosen by solvers). PPI tasks require: (a) Investigating a geometrical figure (from a proof problem) in a DGE (experimenting, conjecturing and testing) to find several new properties of the given figure and related figures that are obtained using auxiliary constructions. (b) Formulating multiple new proof problems. A PPI task is completed only when all the posed problems are solved by the participants.

Figure 3 demonstrates an example of problems posed by Rasha (who studied for a teaching certificate after completion of a B.Sc. in mathematics) who chose the Midline-in-a-Triangle Theorem (see Figure 3, EP is a midline in triangle ABC). To pose new problems she first performed construction of a parallelogram, $EDCB$, used when proving the theorem, and connected different points in the figure.

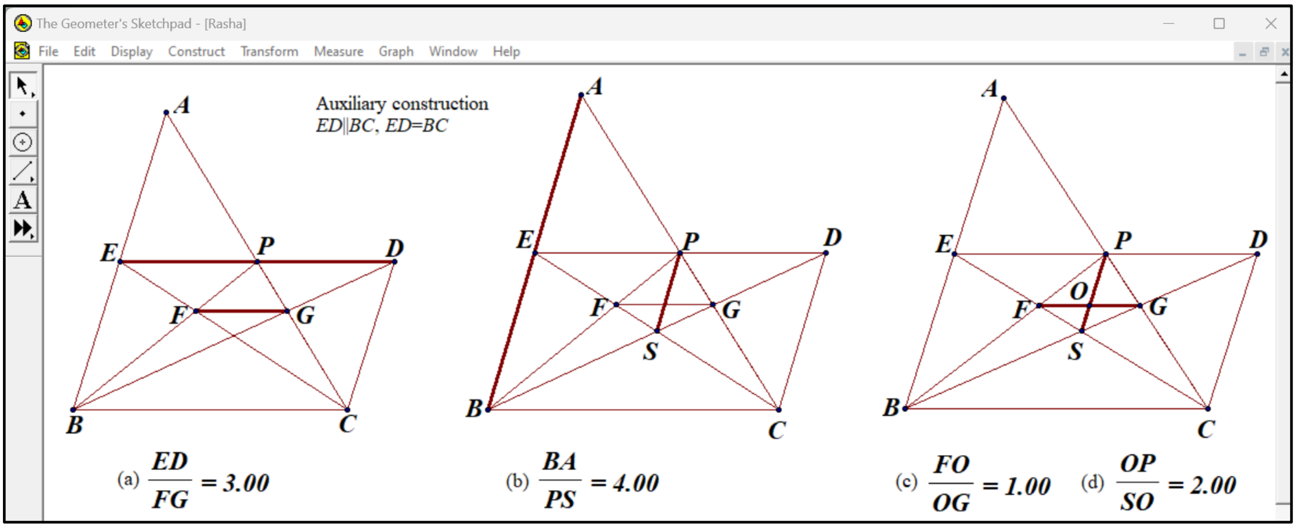

Figure 3: Examples of posed problems

Rasha discovered 4 different properties (depicted in Figure 3): (a) $\frac{ED}{FG} = 3$, (b) $\frac{BA}{PS} = 4$, (c) $\frac{FO}{OG} = 1$, and (d) $\frac{PO}{OS} = 1$, and formulated corresponding proof problems. All the problems were new for Rasha and her peers. These problems did not appear in the textbooks or in the instructional materials available to her.

Like other problem-posing tasks, PPI tasks are open (Pehkonen, 1995; Haylock, 1987; Solver, 1995) since solvers can implement different investigation strategies related to changing/extending the conditions of a given problem through performing auxiliary constructions in DGE, searching for new properties and posing different problems based on the discovered properties. Additionally, PPI openness is related to the differences in the collections of problems posed by different individuals, which include differences in terms of the number, types and complexity of the posed problems. The PPI's openness is also associated with solving a new (posed) problem in that solvers are free to choose how to prove any discovered property. The openness of PPI allows these tasks to be used as a didactical and research tool aimed at the development and evaluation of creativity.

The openness of PPI tasks determines these tasks' complexity, since an investigation can lead in unpredicted directions, conjectures can appear to be incorrect, or solving some posed problems can require knowledge and skills at a level that surpasses the level of problem-solving expertise of those who posed the problems. At the same time, the openness of the PPI tasks and their complexity determines the power of these tasks as tools for the investigation of creativity and problem-solving expertise. The requirement to solve the posed problems links PPI to the participants' problem-solving expertise. Thus, we evaluated both creativity-related skills and proof-related skills linked to participants' performance on PPI tasks (Leikin & Elgrably, 2020; 2022; Elgrably & Leikin, 2021).

The examined creativity components included fluency, flexibility and originality (Haylock, 1987; Silver, 1997; Leikin, 2009). Fluency was defined as the number of investigation strategies used / number of posed problems. Flexibility was defined as the number of different investigation strategies employed/ different posed problems. Originality was defined by the newness and rareness of the investigation strategies / posed problems. The model we employed for the evaluation of PPI is based on the model for evaluation of creativity using multiple solution tasks MSTs (Leikin, 2009).

The examined proof-related components included auxiliary constructions that led to the discovered property, appropriateness of proof and the complexity of the posed problems determined by the conceptual density of the problem (cf., Silver & Zawodjewsky, 1997) combined with the length of the required proof.

Leikin & Elgrably (2020) described an explorative study that examined PPI tasks as a tool for the development of PMTs' proof skills and their creativity components in geometry, and for exploring the relationships between problem-solving expertise and creativity skills. Elgrably & Leikin (2021) explored PPI performance by participants with different types and levels of problem-solving expertise and examined differences in their creativity and problem solving performance when engaged in PPI activity. In Leikin & Elgrably (2022) we made distinctions between outcome creativity (linked to posed problems) and strategy creativity (linked to processes of creation of the new problems through investigations).

The studies demonstrated that both proving skills and creativity components can be developed effectively through employing PPI activities, with significant changes seen in all the creativity components related to the posed problems (i.e., outcome-based creativity). There were no changes in strategy-related creativity linked to engagement with PPI tasks. With regard to types of creativity examined in the study, we found that higher strategy creativity did not necessarily lead to higher outcome creativity, while a higher level of strategy originality correlated with outcome flexibility. Thus creative product and creative process are two distinct characteristics of cognitive processing linked to creativity-directed problem solving.

Focusing on the links between types and levels of participants' problem-solving expertise and PPI we argue that problem-solving expertise at high level significantly influences the quality of PPI as reflected in proof skills and creativity components. Unfortunately, mathematical expertise related to studying mathematics in B.Sc.-level university courses does not affect mathematical creativity linked to PPI. In addition, creativity components of participants with a high level of problem-solving expertise significantly correlated with their proof skills and, moreover, problem posing and proving performed by them appeared to be interwoven. Finally, the role of DGE when conjecturing differed between participants with different levels and types of problem-solving expertise: High level experts tested hypotheses about additional properties and discovered new properties while searching for proofs of complex posed problems, whereas non-experts used a trial-and-error strategy. Correspondingly, high level experts performed auxiliary constructions consciously and with careful planning whereas non-experts used a trial-and-error approach to auxiliary constructions.

REFLECTING ON THE CONSEQUENCES OF USERS' ACTIONS

Reflecting on the consequences of users' actions focuses on a major role of technology as what can scaffold the posing of powerful problems. Often there is a substantial logical distance between the starting points offered by nature and our conjectures about nature and the detailed implications of our models. To help learners make chains of inferences, appropriately crafted software environments can aid dramatically in extending our ability to explore our formal models (Schwartz, 1992). *The Mathematics Imagery Trainer* provides a vivid example in which technology provides and environment in which students explore different perceptual orientations, and these perceptual orientations, in turn, ground prospective mathematical concepts (Abrahamson et al., 2014; Alberto et al., 2021).

In the design of Mathematics Imagery Trainer activities, the *problem* students work on is a motor-control problem—the student is tasked to figure out how to move their body or, by extension, how to operate selected objects in the activity space, such as cursors on a screen, voluminous solids in virtual space, or diagrammatic elements of a unit circle, so as to elicit some designated feedback, for example performing a coordinated bimanual movement that keeps a screen green. As such, Trainer tasks emulate embodied cultural practices, such as swimming, riding a bicycle, or operating a screwdriver—in all these, one must coordinate ongoing sensorimotor actions that maintain a consistent relation between the moving body and the environmental media, whether natural, artifactual, or some combination thereof, so as to perform goal-oriented actions. That is, the *solution* is not static or finite but inherently dynamic—students learn to move in a new way that instantiates the conservation of a mathematical invariant, such as a particular ratio, quadratic function, or sine function, even before they come to realize the mathematical modeling of this new movement form. This design approach implements a theoretical position by which the cognitive activity of engaging in mathematics is not epistemologically different from other cultural–historical practices (Abrahamson & Sánchez–García, 2016; Abrahamson & Trninic, 2015; Shvarts & Abrahamson, in press a, b). This epistemological position is grounded in an evolutionary argument for enactivist mathematics pedagogy, by which the biological cognitive capacity for engaging in mathematical reasoning is an *exaptation* (Gould & Vrba, 1982)—that is, a co-opting—of our species atavistic capacity of perception-for-action (Abrahamson, 2021). Hence, our natural capacity to develop new strategies for perception to guide action (Abrahamson & Mechsner, 2022) is a key idea in our pedagogical development. In turn, key to developing new strategies for perception to guide action is sensorimotor exploration, as we now elaborate.

In Trainer activities, the construct of problem-*solving* is theorized and, therefore, operationalized as pre-symbolic sensorimotor exploration. We view mathematics education essentially as the education of perception (Merleau–Ponty, 2005):

> Any mechanistic theory runs up against the fact that the learning process is systematic; the subject does not weld together individual movements and individual stimuli but acquires the power to respond with a certain type of solution to situations of a certain general form. The situations may differ widely from case to case, and the response movements may be entrusted sometimes to one operative organ, sometimes to another, both situations and responses in the various cases having in common not so much a partial identity of elements as a shared meaning (pp. 164–165)

In a similar guise, Piaget (1971) writes as follow:

> [W]e shall talk of “perception” in the case of a proximate structure of given sensorial evidence; as such, perception can already be seen to intervene in instinctive behavior and to be a no less essential part of kindred behavior. (p. 2)

That is, perception *per se* is an innate general cognitive faculty—the adaptive capacity to organize sensorimotor activity so as to engage the environment effectively. So, when we talk about educating perception, we mean, in fact, constructing new neural substrates by which perception can govern sensorimotor activity that is effective to accomplish some task of cultural significance. For example, when we learn to ride a bicycle, we are building the cerebral networks that let our perception respond rapidly to emergent environmental contingencies by activating sensorimotor behaviors that keep us moving stably—we develop what Bernstein (1996) called the *automatisms*. Importantly, educating perception is never about somehow improving the sensory organs or the muscles themselves but figuring out new Gestalts as mental structures that govern automatic sensorimotor activation (Mechsner 2003, 2004; Mechsner et al., 2001). In summary, we look to create learning experiences that emulate enactivist epistemology:

> In a nutshell, the enactive approach consists of two points: (1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided. (Varela et al., 1991, p. 173)

Whereas Trainer activities appear similar to certain Dynamic Geometry Environments, such as GeoGebra (Leung et al., 2013), they also differ from these DGE modules. In Trainer activities, the objects that students operate are not computationally pre-constrained to maintain the targeted mathematical relations, for example, Trainer students can position the objects in a configuration that violates the targeted conceptual instantiation. In Trainer environments, only the system's feedback will indicate whether or not the student's proposed configuration abides with the hidden rule. Thus, Trainer students need to discover the rule and self-impose it as a new constraint on their sensorimotor actions: they assimilate the discovered rule and accommodate their sensorimotor behavior accordingly. That is, in Trainer environments the constraints are implicit to the feedback, so that students need to figure out what they may or may not do in order to receive the favorable feedback, whereas in classical DGE the constraints are inherent to the manipulability of the objects themselves, so that students can't help but operate within the constraints regardless of how they manipulate the objects. By way of gross analogy, Trainer activities allow you to fall off your bicycle as you learn to ride it, whereas classical DGE activities never remove the training wheels. Abrahamson and Abdu (2020) draw on the literature of the movement sciences to argue for the pedagogical advantages of Trainer "open" tasks (oDGE), where students need to discover and self-impose constraints on their perception–action loops, as compared to "closed" tasks (xDGE), where the constraints on action are preconditions of the interactive system. The relative advantages of these two approaches remain to be evaluated empirically.

Teachers or, more generally, human or AI pedagogical agents, can play active roles in guiding students' sensorimotor engagement and mathematical sense-making in enactive learning environments (Abrahamson et al., 2012). Using multimodal semiotic resources—words, gestures, diagrammatic structures, etc.—teachers introduce constraints suggest affordances that modify the student's engagement with the environment (cf. Churchill, 2022; Newell & Ranganathan, 2010). By way of analogy, a piano teacher might place coins on their student's hands and ask that they continue playing without dropping the coins. These interventions in the *micro-zone of proximal development* orient students toward elements of the environment whose perception could increase the students' grip on the world, ushering in normative cultural practice (Shvarts & Abrahamson, 2019). Next, teachers introduce supplementary mathematical instruments into the working space, such as a grid of lines, and create opportunities for students to discover how these new resources may serve as means of extending their existing sensorimotor activity.

Students discern in these new resources features affording potential utility for enhancing either the enactment, evaluation, or explanation of their strategy. Yet in the course of instrumentalizing these digital artifacts to improve on their existing task performance, students implicitly appropriate the cultural–historical practices potentiated by these epistemic forms, and consequently their strategy is modified, often from qualitative to quantitative procedures (Abrahamson et al., 2011). Teachers facilitate this semiotic process: they “re-voice” students’ gestures and speech to highlight, objectify, elaborate, and stabilize students’ mathematical interpretations of their own sensorimotor engagement (Flood et al., 2020). As such, the student–teacher dyad establishes mathematical notions through negotiating for the mutual intelligibility of their respective and joint actions (Flood, 2018).

THE IMPACT OF DIGITAL ENVIRONMENTS’ ON LEARNERS’ MATHEMATICAL KNOWLEDGE AND THE ROLE OF THE TEACHER

Students, and to some point their teachers, would use a pragmatic set of criteria for assessing intellectual worth, making sure knowledge of a particular piece of subject matter be turned to advantage in the outside world in which one lives. Yet Schwartz (1992) notes also that “people are engaged by more than just the pragmatic. They are often engaged by interesting complexity, particularly if it is complexity of their own making”. We can think of environments as offering people the opportunity to fashion and explore complex situations in domains that our culture has come to regard as important, e. g. Seeing the Entire Picture - STEP (Olsher et al., 2016). However, it is well-documented that many teachers are challenged to support learners to engage productively in such environments, when most have never previously experienced authentic intellectual progress alongside technological tools (Clark-Wilson & Hoyles, 2017; Noss & Hoyles, 1996).

Since the early research on digital technologies in the 1980s and 1990s, which brought the role of the teacher into view, it is well-documented the introduction of epistemically rich technologies to school mathematics is influenced in a number of ways which include the teachers. One aspect is the prior experiences as learners of mathematics with, or without access to digital tools. The fact that many digital tools are available for many decades, does not imply that current teachers had a chance to use them as students. When teachers have hands-on experiences of the digital tool to work on the (successive) problems that they will subsequently offer to their students, it impacts the teacher’s own knowledge and practice.

More is now known in some countries about the nature of the more impactful professional learning opportunities that pre- and in-service teachers engage with and value. Furthermore, as such professional learning opportunities also require careful design and implementation, which requires a group of expert others, who might be known as champions, or mathematics teacher educators. Research on the nature and role of professional learning communities within the specific context of student technology use for the purpose of problem solving is also emerging, with some promising findings that can inform new country, school and classroom contexts. In addition, it is well known that teachers' personal perspectives with respect to school mathematics curricula and teaching approaches have a great influence on their practices. Furthermore, their views of technology may align or conflict with these ideas. Writing this in 2023, we find ourselves in the midst of a global discourse on the impacts of artificially intelligent mathematics tools on that are rapidly negating the need for rote learning of traditional numeric and algebraic algorithms. This negates the type of problems that are present in most high-stakes assessments of school mathematics, and leaves wondering whether their new role will be to restrict students' access to digital technologies, or join the widening lobby to radically review school learners' educational experiences, and the place of mathematics within it. Furthermore, there are a number of institutional factors that also influence how teachers can come to know, develop, and sustain the student use of technology within more problem solving contexts. Factors that are known to increase the likelihood of teacher growth in this area include: Supportive national, regional and local mathematics curricula; mathematically- and pedagogically-aligned assessment methods and practices (especially high-stakes testing); high quality professional learning activities, resources and communities, which are designed and supported by suitable experts; and supportive national, regional and local policies and resourcing for digital tools.

The mathematics education research literature abounds with examples of small-scale, mostly exploratory studies that conclude promising findings with respect to teachers' motivational, confidence and epistemic growth within the context of student problem-solving with technology. However, few studies are followed up by larger-scale research initiatives that aim to understand effectiveness (and possibly efficacy) of such approaches in multiple schools and classrooms. This is a problem for the field, as in the real world of schools globally, there are multi-millions of free and paid-for digital educational resources available, some of which will be positioned towards problem solving in mathematics education. Although, these resources compete with the more research-informed resources, they are rarely accompanied by support resources (material and human) that can enable them to be implemented widely and with increasing effectiveness in diverse classrooms. However, the research field is mature enough to be able to propose more research-informed professional learning opportunities for teachers on a wide scale - enhanced and informed by technology itself.

One way to tackle the challenges in the pursuit of interesting mathematical knowledge in a mathematical classroom is demonstrated by design principles of the STEP platform. Beyond providing the students with rich tasks that enable them to express their own mathematical ideas using learner generated examples and explore different mathematical concepts through different patterns of example eliciting tasks (Olsher, 2022), STEP demonstrates means to increase the interaction with mathematical knowledge harnessing also digital tools for analysis of student work.

STEP analyzes student answers as mathematical objects. This analysis creates a set of automatically assessed characteristics for student's answers. For example, Abdu et al. (2021) presented a set of characteristics that were assessed for quadratic functions submitted by students as examples for quadratic functions that their graph passed through two given points. While the functions were assessed also for whether their graph indeed passed through the given points which indicate whether the answers was correct or not, other characteristics were also automatically assessed. Among these characteristics were the type of extremum point (minimum or maximum), the number of x-axis intercepts (zero, one or two), the form of the expression (polynomial, vertex or intercepts). While these characteristics are not critical for the correctness of the answer to the task, they provide the teachers and the students an accessible means of interaction over different mathematical ideas related to the task at hand, providing flexibility in terms of the mathematical knowledge that takes form in the interaction over the task. On a student level, one can learn to communicate about different mathematical objects in various ways either during the work on the task itself or after submitting it (Olsher & Thurm, 2022; Yerushalmy, Olsher, Harel & Chazan, 2022).

This analyzed data about student work in used to create different interactive tools that make different aspects in students work more accessible for the teacher. Using the different interactive visualizations, the teachers can choose whether they want to manually browse through their student's work, and drill down into specific students' work. Teachers can also further their analysis using statistical information, as well as interactive filtering tools resembling online shopping websites to filter student's work and inspect characteristics they find as relevant, as well as their relation to other characteristics (Abu-Raya & Olsher, 2021). In their use of STEP in their classrooms, we observed that teachers shift the focus of their teaching from mostly student mistakes to other characteristics of student answers (e. g. interesting non critical characteristics of student answers), thus providing an example of flexibility in what they find as relevant, worthwhile, mathematical knowledge (Olsher & Abu- Raya, 2019).

CONCLUDING REMARKS

In this paper we take a contemporary perspective to revisit the interplay between technology and conjecturing within the process of problem-solving. This perspective, rooted in sub-themes conceptualized over three decades ago, is demonstrated through the use of up-to-date technologies. The manifestation and evolution of the "classic" ideas and categorization provide a unique opportunity to value both the fundamental ideas in combining technology in conjecturing, and also the influence of the current developments on the evolution of the ways we perceive problem solving over the years.

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THEORY AND PRACTICE OF DESIGNING EMBODIED MATHEMATICS LEARNING

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Different approaches to embodied learning—conceptual learning of curricular content grounded in a new capacity for enacting forms of purposeful physical movement in interaction with the environment—have become increasingly central to mathematics-education research. This research forum provides participants with an up-to-date overview of diverse and complementary theoretical perspectives on embodied learning, principles derived from these perspectives governing the design of environments for learning various mathematical content, and demonstrations thereof. We speculate on promising directions for future embodied design research.

INTRODUCTION

Embodied mathematics learning is grounded in the human capacity to interact with the physical and social environment through purposeful movements, use of the senses, and creation and utilization of artifacts. As a field-wide paradigm, embodied learning stems from the embodied turn in the cognitive sciences, which maintains that perception and action are formatively constitutive of our thinking—cognition is inherently modal and situated activity (e.g., Chemero, 2013) that draws on the body's physical interaction with the world (Gibson, 1986). As such, cognition, including learning and knowing, emerges from activity of the perceptual and motor systems and, thus, in turn, is shaped by the body's physical properties and movement capacity (Glenberg, 2010). These ideas provided a powerful incentive for updating curricula design and resources for learning and teaching (Shapiro & Stolz, 2019).

There are several pedagogical precursors of the embodied approach to learning. Among them are Friedrich Fröbel's ideas on the importance of the child's activity in learning (Brosterman, 1997), Dewey's (1938) conceptualization of 'learning by doing' and reflective inquiry, and Maria Montessori's (1949) work based on the observation that "Watching [the child], one sees that he develops his mind by using his movements….[One cannot learn by] sitting down, moving nothing" (pp. 203–204). Similarly inspired early researchers of mathematics education advocated an educational approach capitalizing on the importance of physical space for learning: "We live in space, …. move in space, …. analyze space, to be better adapted" (Freudenthal, 1971, p. 418). This Research Forum (RF) also heeds Schoenfeld's (2016) statement that the variety of approaches to embodied learning (see Abrahamson et al., 2020) is becoming increasingly important for mathematics

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 159-189). PME 46.

education. The RF provides an up-to-date overview of diverse yet by-and-large complementary theoretical perspectives on embodied learning, including their philosophical roots and influences from cognitive science. The exposition continues with design principles derived from the theory, then examples of environments designed by the contributors to this RF for learning various mathematical content. We conclude with perspectives and future directions for research on embodied learning in mathematics education.

THEORY OF EMBODIED LEARNING OF MATHEMATICS

A tree of embodied perspectives

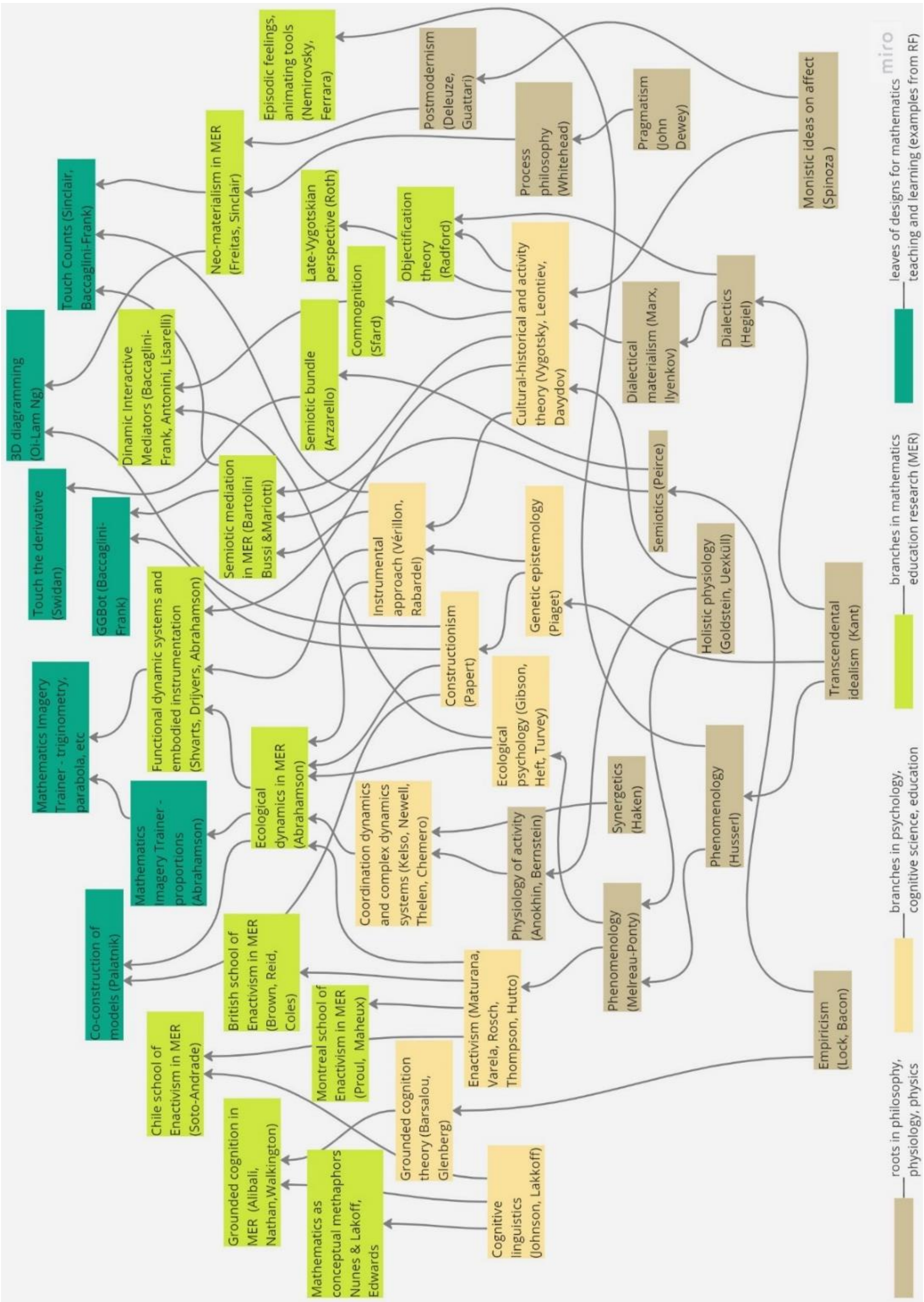
There is a great variety of theoretical approaches in mathematics education research that call for attention to bodily processes that enable mathematical thinking and learning, including, but not limited to motor performance, gestures, and eye movements as well as multimodal sensory experiences. The scope of these approaches is difficult to discern, as they are rooted in diverse ideas from philosophy, biology, physics, and branches of cognitive science and psychology, including cognitive linguistics, developmental psychology, the science of movement, and many other disciplines (see Figure 1 for a rough sketch of the tree of ideas).

Traditionally, researchers distinguish between conservative and radical approaches based on their stance towards mental representations and the separation between mind and body (Hutto & Abrahamson, 2022). For example, *grounded cognition* (Barsalou, 1999, 2021) assumes that a variety of experiences gained through eyes, ears, and other body parts is accumulated in mental representations called *perceptual symbol systems*. Grounding mathematical concepts on those experiences allows researchers to theorize the role of gestures and other forms of embodiment (Nathan & Alibali, 2021; Walkington et al., 2022). Another relatively "conservative" approach comes from cognitive linguistics: Lakoff and Johnson (1980) introduced the cognitive semantics theory *of conceptual metaphors* to explain how language propagates embodied experience into abstract concepts. Metaphorical mapping of mathematical concepts on bodily experiences, such as treating sets as containers, was extensively explored by Lakoff and Núñez (2000) and spread through math-ed scholarship (e.g., Sfard, 1994). "Mapping" between different contexts, e.g., written numbers, a number line, and physical objects, would form conceptual integration (Edwards, 2009). Once such mapping and grounding are assumed, educators might wish to induce mathematical understanding by inviting students to gesture out particular shapes hypothetically associated with mathematical content (e.g., Walkington et al., 2022).

Radical approaches abandon the idea of conceptual structures as ecologically independent epistemological entities preserved by the cognitive system as representations or schemes, instead embedding concepts into embodied experiences (Varela et al., 1991). A key root of those approaches is *complex dynamic systems* and *coordination dynamics* ideas (Bernstein, 1967; Kelso &

Schöner, 1988), which propose the non-linear and self-organizing nature of cognitive processes. Another intellectual grounding of embodied learning is *ecological psychology* (Gibson, 1986), which states that an individual's perception constantly evolves by actively detecting in the environment aspects relevant to interaction, i.e., *affordances*. While the interrelation between these approaches is hotly debated in cognitive science (Di Paolo et al., 2021), their combination is productive for theorizing mathematics learning (Abrahamson & Sánchez-García, 2016) and designing for learning within the embodied design framework (Abrahamson, 2014).

Figure 1: A tree of embodied perspectives.
https://embodieddesign.sites.uu.nl/tree/



Diving into the mathematics education research field, we may notice that some embodiment-oriented approaches bypass cognitive science; thus, the radical versus conservative landscape does not apply to their classification. For example, the *neo-materialist* approach (de Freitas & Sinclair, 2013) is inspired by Deleuze's ideas of a concept as an *assemblage*, where material artifacts, nature, and bodies get intertwined in their actualization. This approach calls for attention to materiality and how it invites students to interact. Grounded in Hegel's and Vygotsky's ideas, the *theory of objectification* (Radford, 2021) considers cognition to be sensuous and explores how students become aware of mathematics as they encounter material culture in practical collaborative activity with teachers and peers.

When theoretical tenets rest on disparate ontological and epistemological assumptions and share only family resemblance, still they may nevertheless undergird common pragmatic agendas. For example, researchers who conceptualize a cognitive mapping between sensorimotor experiences and mathematical ideas (Walkington et al., 2022) may embrace principles of *ecological dynamics* (Abrahamson & Sánchez-García, 2016). Another combination comes from joining enactivist ideas with metaphorical mapping between mathematical concepts and other interaction contexts (Díaz-Rojas et al., 2021). Yet another approach combines coordination dynamics and cultural–historical ideas into the notion of a *functional dynamic system* (Shvarts & Abrahamson, in press). In this monist approach, mathematical notions are reconsidered as direct extensions of the bodies via mathematical artifacts that come forth in the intercorporeal sensorimotor dynamics of teachers' and students' task-oriented embodied activity.

In the following sub-section, we present the key ideas essential for understanding mathematical learning as an embodied process: moving in a new way, perceiving in a new way, and naming in a new way. We do not offer a coherent theory but point to the ideas of many authors who, using different terms, refer to these aspects of learning. We strive to bridge these perspectives around the same phenomena.

Moving in a new way

Movement is the deepest aspect of human nature:

> … we are essentially and fundamentally animate beings. In more specifically dynamic terms, we are animate forms who are alive to and in the world, and who, in being alive to and in the world, make sense of it. We do so most fundamentally through movement, unfolding a kinetic aliveness that is in play throughout the course of our everyday lives from the time we are born to the time we die. (Sheets-Johnstone, 2011, p. 452)

When learning and developing, human bodies come to move in new ways: sucking, walking, writing, drawing. To grasp how understanding is grounded in movement, we must acknowledge that body movement already requires understanding the world. An alive movement is not a blind repetition of a pre-programmed sequence of motions. Instead, it is an emergent phenomenon that arises in a constant attempt to solve motor problems that the world places in front of an organism (Bernstein,

1967) as it strives for relative equilibrium (Merleau-Ponty, 2002) in a constantly transforming environment. Many levels of regulation act collaboratively in accomplishing a movement: Our body responds to gravity, our posture is built in relation to all body parts, and our hands encompass the spatial relations of the objects (Bernstein, 1967). All these levels find their way through constant probing and adjusting of sensory-motor processes by anticipating and receiving feedback from the environment. An exact repetition is never possible in this complexity. Yet, invariants are enabled by the synergetic character of our motor systems (Kelso & Schöner, 1988): a multiplicity of muscles assemble into a functional dynamic system that enables a relatively stable movement that repeatedly and efficiently solves motor problems (Bernstein, 1967). To get a grip on this theoretical idea, stand for a while on one leg and observe a complex play of the muscles of your supporting leg: never stable, in continuous co-adjustments, they enable your continuous still position. Try then to swing your free leg. You may notice how your posture became shakier for a while and then regained a relatively stable balance, continuously supporting your swings by iteratively modifying the position of your mass center.

Perceiving in a new way

Through "synergies of meaningful movements" not only the motion but the world itself is constituted for the subject (Sheets-Johnstone, 2011, p. 453). Just like the muscles' activation is organized into synergies, our sensors organize into meaningful perceptual experiences: we come to discern from the world what is relevant to our enactment. *Attentional anchors*—namely, perceptual structures (Gestalts) that arise to serve enactment or are given by educators to support new forms of action—act as a proxy between the world and the actor (Abrahamson & Sánchez-García, 2016). Notice, when swinging your leg, you likely stopped being focused on the muscles of your supporting foot but paid attention to the swing trajectory of your free leg— imbricating perceptual structure on the world.

A focus on the living body moving in the environment suggests that the environment is not independent of the perceiver but provides *affordances*—possibilities to act (Gibson, 1986). Perception of affordances is direct but not innate. On the contrary, it constantly develops as a learner discovers higher-order variables through differentiating relevant information. It is through movement in the environment that learners can come to perceive "aspects of stimulus information that persist despite movements of the perceiver (or are actually brought into existence by those movements), and that correspond to relatively permanent features of the objective situation. (Neisser, 1987, p.12)

Applying those ideas to Dynamic Geometry Environments, a figure can be conceptualized as a set of affordances that the solver comes to perceive through dragging. Further, we may expect that, through dragging, a learner will discover new mathematical *invariants* (Leung et al., 2013). Similarly, Mason (1989, 2008) characterizes learning by *what (focus)* is attended to and *how (form)* the objects are attended to. He distinguishes five different forms of attention: *holding the whole* without focusing on particularities or *discerning details* among the other elements of the attended object. From there, one may *recognize relationships* between discerned elements, *perceive properties* by actively searching for additional elements fitting the relationship, and, finally, *reason based on perceived properties*. Mason's theory has been applied by Palatnik (2022) to the analysis of embodied activities for learning spatial geometry—he identified correlations between students' physical interactions with artifacts and their attentional shifts to inherent properties and structures.

Finally, Marx introduced the idea of "sensuousness as practical activity" (Radford, 2021), allowing a further understanding of human perception as shaped by cultural practices within society. Accordingly, Vygotsky considers perception to be a higher-order function—a complex systemic social entity that cannot be reduced to sensation *per se* (Vygotsky, 1978). Human practices in specific domains, such as archeology, reveal *professional vision*: an ability to notice structures that are unnoticeable by a non-experienced eye (Goodwin, 1994). In like vein, Radford (2010) discusses the role of *theoretical perception* in mathematics: patterns that educated adults perceive, students must learn to distinguish. Teachers appropriate multimodal resources, such as gesture and rhythm, to highlight those patterns in the environments (Goodwin, 2018; Radford, 2010). To make sense of teachers' rich utterance, students actively scan the environment to establish concordant perception (Roth & Thom, 2009; Shvarts, 2018). This is why designing special *fields of promoted actions* might be beneficial for learning to move and perceive in a new way (Abrahamson & Trninic, 2015).

Naming in a new way

While new ways of moving and perceiving are the key parts of embodied mathematics learning, embedding those individual—often idiosyncratic—dynamic practices into cultural discourse and environments requires further theorization. In embodied design, learning might start by developing new sensorimotor coordination or individual exploration of an artifact that triggers new forms of perception and only later becomes gradually interconnected with more and more advanced cultural artifacts (Abrahamson et al., 2011) and scientific discourse (Flood, 2018; Mariotti, 2009). Importantly, artifacts and discourse are not self-evident for students. While mostly rooted in Vygotskian ideas, theories vary in explaining how artifacts and discourse come to be part of students' mathematical knowledge

Theory of Semiotic Mediation (TSM) argues that the relation between the artifact and the learners in the course of accomplishing a specific task is expressed by signs such as speech, gestures, symbols, and tools (Bartolini Bussi & Mariotti, 2008). On the one hand, when repetitively accomplishing a (well-designed) task with an artifact, a solver develops schemes (Vergnaud, 2009), which constitute a "hidden" psychological component associated with the visible actions and with the other signs produced. An artifact and a scheme constitute an instrument—a psychological construct used for further instrumental actions (Verillon & Rabardel, 1995). On the other hand, while solving the task and interacting with peers, a learner produces personal signs that are closely related both to the task and to the artifact, thus constituting personal meaning. The *semiotic potential* of the artifact with respect to the mathematical knowledge allows the teacher to conduct mathematical discussions involving a variety of signs (e.g., Mariotti, 2009). In these discussions, the signs produced by the students (artifact signs) are gradually transformed into shared signs. Such shared signs generalize the situated signs referring to personal meanings, and are transformed into mathematical signs (through successive didactical cycles) to the knowledge being taught.

Other theories avoid talking about meaning and scheme as independent cognitive constructs, thus removing representationalist ideas from mathematics education discourse. Following the *commognition* perspective (Sfard, 2008), sensorimotor processes comprise a mathematical object's realization tree, consisting of various inscriptions, such as formulas, sketches, definitions, and primary objects of the world interconnected. From a *functional dynamic systems perspective* (Shvarts et al., 2021), artifacts and discourse directly extend the dynamics of students' bodies once they are appropriated for solving the task. The students' ways of acting and naming come to correspond the cultural forms of acting and naming through joint-attention with a teacher, when a student's and a teacher's sensory-motor processes are coordinated in common environment (Shvarts & Abrahamson, in press).

Overall, cultural artifacts and discourse transform students' perception by making it mediated by the artifacts (Bartolini Bussi & Mariotti, 2008; Vygotsky, 1978) and allows them to decrement conceptual aspects and see mathematical properties (Leung et al., 2013; Mason, 2008). Soon, though, perception accommodates from *mediated* to *immediate*, as the artifacts seamlessly "stick" to the bodily system, giving rise to new forms of concrete experience (Shvarts et al., 2022). Thus, artifacts, including discourse, appear to be new tools for actions and lenses for perception: a student equipped with a ruler sees distances as measurable; or, equipped with a notion of exponential growth, she readily anticipates the shape of a graph depicting epidemic spread of a disease (cf. Vérillon & Rabardel, 1995, on utilization schemes).

DESIGN PRINCIPLES OF EMBODIED LEARNING

The design principles (DPs) underlying artifacts used in embodied learning are motivated by the perspective that human beings think with and through their body (Merleau-Ponty, 2002; Radford, 2021). This perspective has stimulated many design-researchers to create artifacts fostering bodily engagement in mathematics learning. This section, organized along five DP, outlines tenets and heuristics shared by all the RF authors.

DP1: Involve students' bodies in the learning process

This involvement can be achieved through perception-based design artifacts or action-based design. The first genre builds on learners' early mental capacity to draw logical inferences from the perceptual judgment of intensive quantities in source phenomena; the second genre builds on learners' perceptuomotor capacity to develop new kinesthetic routines for strategic embodied interaction (Abrahamson, 2014).

The dualistic view of cognition, which distinguishes between the mind and the body, considers our body movement (e.g., gestures and gaze) as a window through which we can make inferences about the learner's perception and thinking. On the contrary, the monistic view of cognition, which considers the unity of the body and mind, considers our body movement a constitutive part of our thinking, because the use of any kind of artifact creates a specific kind of interaction and, thus, a specific kind of thought. Hence, our second principle for embodied environment design:

DP2: Offer immediate sensorimotor interactions with artifacts

Perception is viewed as a cognitive structure that emerges first as the psychological means of enabling some stable form of coordinated motor engagement with the environment, then as the reified semiotic kernel of mathematical practice, including verbal and nonverbal language, gestures, extra-linguistic expression, and inscription (Abrahamson & Mechsner, 2022). These semiotic means give rise to learners becoming aware of mathematical knowledge embedded in the artifact (Bartolini Bussi & Mariotti, 2008; Radford, 2021). Hence, our third design principle is

DP3: Attend to the semiotic sensitivity of the design

Semiotic sensitivity considers the signs included in the artifacts, which can be produced by the educators or by the students through solving a task. Designers should deeply reflect on the genre of signs and their relationship with mathematical knowledge, which they expect to connect through the activities with the artifacts. They should also reflect on the ways students will interpret and endow with meanings the several genres of signs. For example, the designer should be sensitive to the colors used in the artifact and the potential associations between these and the target mathematical knowledge. Mathematical understanding happens when the learners are able to convert from one semiotic register (representation) to another and to treat within the same semiotic register (Duval, 2006). Thus, our fourth principle:

DP4: Include a variety of semiotic registers and artifacts that potentiate mathematical perception and discourse

Gallese and Lakoff (2005) claim that sensory-motor system of the brain is multimodal and theorize language as multimodal as it includes speech, gestures, drawings. Thus, to understand cognitive processes we should analyse all the modalities (Arzarello & Robutti, 2010, Abrahamson et al, 2020). The embodied-design approach puts forward that learning new concepts begins with discovering new ways of acting in the environment (Abrahamson & Bakker, 2016). While sprouting from enactment, embodied activities further inevitably require socially scaffolded “languaging” to enter cultural discourse (Flood, 2018). Multimodality lies at the core of learning. Thus, our fifth design principle is:

DP5: Foster multimodal engagement and “languaging”.

DESIGN OF EMBODIED LEARNING FOR THE DIFFERENT MATHEMATICAL TOPICS

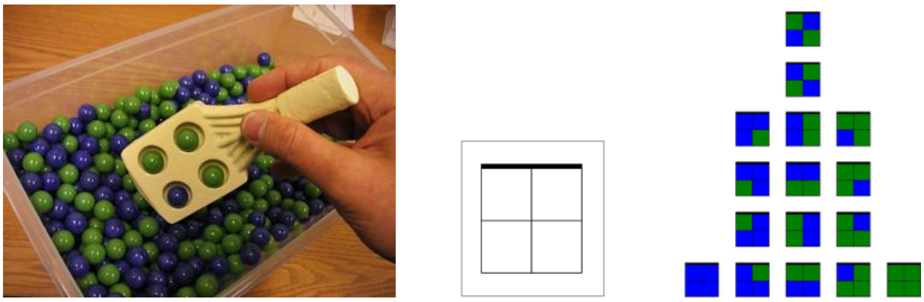
Embodied design: A framework in search of a theory

Originally emanating from a researcher’s practical experience as a mathematics tutor and resource innovator, *embodied design* (Abrahamson, 2014) gradually sprouted theoretical roots. These roots wandered among different grounds, at first testing “conservative” forms of embodiment, such as the *cognitive semantics theory of conceptual metaphor* (Lakoff & Núñez, 2000), *conceptual symbol systems* (Barsalou, 1999), and various intellectual foundations of gesture studies, such as *gesture as simulated action* (Hostetter & Alibali, 2008). However, micro-ethnographic analyses of data collected in empirical evaluations of embodied-design activities were raising challenges to the basic epistemological and ontological assumptions of those theories. Moreover, Dor Abrahamson was affiliated with Uri Wilensky’s Center for Connected Learning and Computer-Based Modeling, where he had developed computationally enabled projects to foster youth understanding of natural and social phenomena from a complexity perspective, and so, Abrahamson had espoused the methodologies of dynamic systems theory as his *modus operandi* in modeling the ontogenetic emergence of conceptual understanding in socio-cultural contexts (Tancredi, Abdu, et al., 2022). Still wandering, embodied design drew on movement science (Abrahamson & Mechsner, 2022), evolutionary biology (Abrahamson, 2021), dance scholarship (Abrahamson & Shulman, 2019), and contemplative practice (Morgan & Abrahamson, 2016). Increasingly radicalized, though, embodied design turned to *ecological psychology* (Gibson, 1986), *coordination dynamics* (Kelso, 2000), and their amalgamated *ecological dynamics* (Araújo et al., 2020), and then further “left” to enactivism (Hutto & Myin, 2013), which led to collaborative scholarship (Abrahamson & Sánchez–García, 2016). While embodied design is fairly grounded now, it keeps searching for nutrients that will increase the integration, coherence, and generalizability of its theoretical models. Currently the work looks to elaborate on Vygotsky’s cultural–historical theory of cognitive development using phenomenological and complexity tools (Shvarts & Abrahamson, 2019, in press). Below we exemplify two genres of activities emanating from the research program, *perception-based* and *action-based* embodied design (Abrahamson, 2014).

Perception-based genre of embodied design: The Seeing Chance project

Humans have innate or early-developed perceptual sensitivity to ecologically adaptive exemplars of *intensive quantities*, that is, quantities that are scientifically defined as *a/b*, for example, velocity (distance / time), aspect ratio (height / width), or likelihood (favorable events / possible events) (e.g., Xu & Garcia, 2008). We can perform judgments on these *perceptually privileged intensive quantities* (Abrahamson, 2012), such as determining the representativeness of color samples from a mixed-color population, just as long as the quantities are presented asymbolically (cf. Zhu & Gigerenzer, 2006). Yet such tacit perceptual capacity by no means implies explicit mathematical knowledge. To the extent that educators wish to leverage students' tacit capacity as a cognitive grounding for conceptual understanding, further design resources and mediation techniques are required to coordinate the natural and cultural. Abrahamson (2012c) sought to leverage tacit judgments of likelihood as an epistemic grounding for the notion of probability.

Figure 2: Selected materials from the Seeing Chance design for the binomial. Project page: https://edrl.berkeley.edu/projects/seeing-chance/



a. b. c.

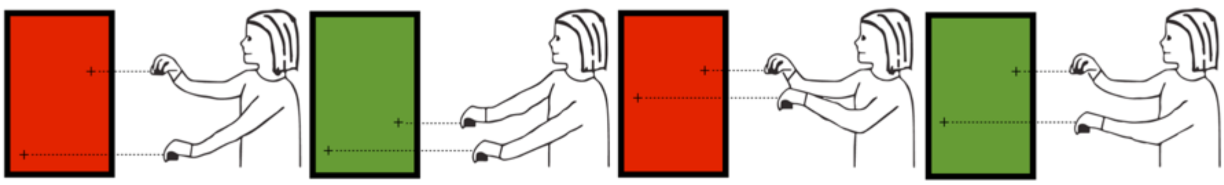
The *marbles scooper* (Fig. 2a) is a random generator approximating a binomial experiment. Bearing 4 concavities, the scooper draws samples of 4 marbles from a tub containing equal numbers of green and blue marbles. Young students correctly predict that the most likely outcome is a scoop with 2 green marbles and 2 blue marbles, the least likely scoops are of uniform color, etc. However, students do not attend to the specific *order* (pattern) of the marbles sample, only to the green-to-blue *ratio,* which they compare to the green-to-blue ratio in the tub. Students then use a set of empty 2-by-2 iconic formats of the scooper (Fig. 2b) and green and blue crayons to create "all the things we could get when we scoop." The tutor guides them to create not just five cards (e.g., the bottom row of Fig. 2c) but all 16 permutations. Once the entire sample space is completed and configured, students spontaneously realize how to perceive it as a model of the source phenomenon that enables them to argue for their initial judgment. This negotiated learning sequence has been called *product before process* (Abrahamson, 2012b), where students perform a *semiotic leap* (Abrahamson, 2009) via *abductive reasoning* (Abrahamson, 2012a) that bridges the natural–cultural gap.

Action-based genre of embodied design: The Mathematics Imagery Trainer

Varela et al. (1991) submit that “(1) perception consists in perceptually guided action and (2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided” (p. 173). The Mathematics Imagery Trainer is an *instrumented field of promoted action* designed to elicit recurrent sensorimotor patterns from which emerge cognitive structures grounding targeted concepts, such as proportion (Abrahamson, 2019).

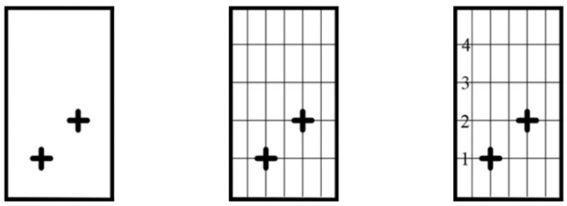
Figure 3 features a Mathematics Imagery Trainer for proportion. The student’s task is to place the cursors at locations that make the screen green. Unknown to the student, the screen is green when the respective heights of the two cursors over the screen base relate at a set ratio, here 1:2. Through exploration (Fig. 3a), the student happens to make the screen green (Fig. 3b). Asked to move her hands continuously, keeping the screen green, she keeps the interval between the hands fixed, thus violating the ratio (Fig. 3c). With time, she realizes that the interval should change with the height.

Figure 3. The Mathematics Imagery Trainer for Proportion—the Parallels motor-control problem: schematic depiction of four key events in learning to enact the conceptual choreography. https://edrl.berkeley.edu/projects/kinemathics/



a. b. c. d.

Figure 4. Three schematic interface modes of a Mathematics Imagery Trainer for Proportion—the Parallels motor-control problem



a. b. c.

Having worked in continuous space (Fig. 4a), we then introduce a grid on the screen (Fig. 4b). Students appropriate this artifact as a pragmatic-cum-semiotic resource, enhancing the enactment, explanation, or evaluation of their method, shifting into math discourse. Once we introduce numbers (Fig. 4c), students draw on their multiplicative knowledge to re-describe their actions. So doing, they are able to coordinate logically between their various strategies (Abrahamson et al., 2011, 2014). At the same time, classroom observations reveal that appropriations of a grid or numbers might be problematic for some students (Alberto et al., 2022).

Instead, children might prefer to use self-invented artifacts, such as rulers or dice, for marking the length and measuring the ratio. As Palatnik and Abrahamson (2018) found, in the absence of a grid, students may utilize the cursor icons both for measuring and to develop rhythmic forms of movement that facilitate task performance.

Moving action-based embodied design to more advanced mathematical topics: parabola, trigonometric functions and statistics

The domain of functional relations appeared to be a natural area for further exploration of the action-based design genre. Graphs are the most usual way to represent functions, however, perceiving a Cartesian plane requires special forms of perception (Krichevets et al., 2014; Radford, 2010), avoiding focusing on irrelevant aspects of a graph (Arcavi, 2003). The action-based embodied design promotes the development of new forms of motor action and, thus, new forms of perception (Abrahamson, 2019). For example, a mathematical perception of a parabola would include seeing the curve as the set of points that are equidistant from a directrix and a focus. In our action-based embodied activities for parabola, students play with a triangle's vertex and discover that triangle is green when it is isosceles, and its vertex is equidistant from a line and a point (Shvarts & Abrahamson, 2019, Figure 5, a, b). In further mathematization, they interconnect the trace of the isosceles triangle's vertex with a quadratic equation (Figure 5 c).

Figure 5. Action-based embodied activities for learning parabola. The triangle turns green when AC=AB. The letters and the parabola were not visible to the students.

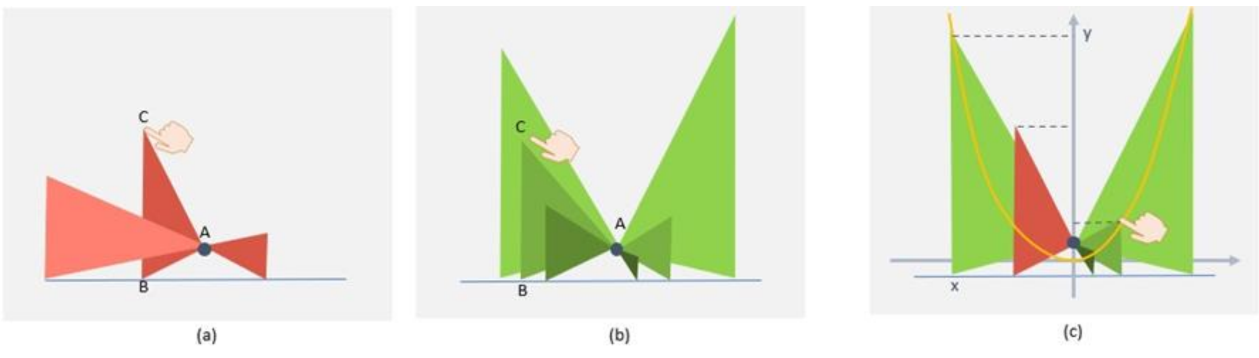
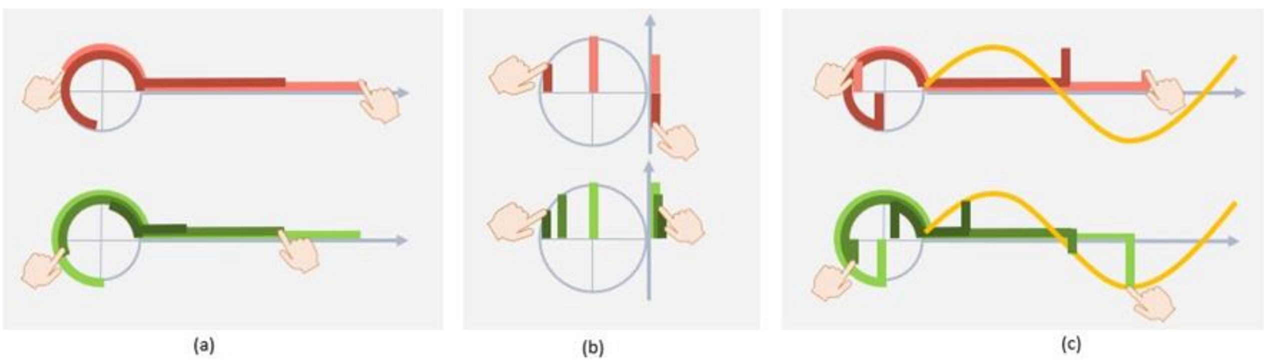


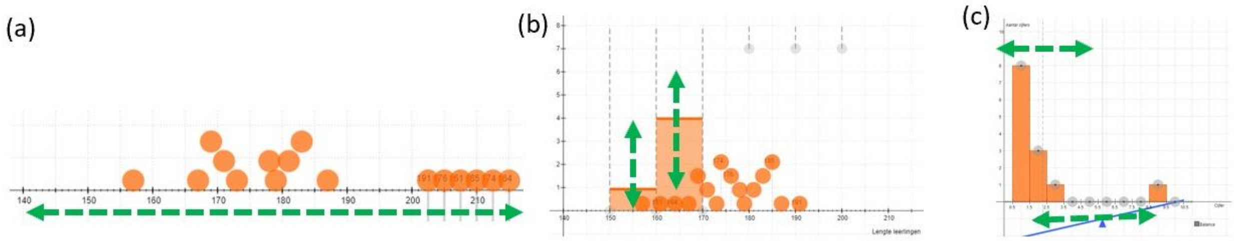
Figure 6. Action-based embodied activities for learning trigonometry.



Learning trigonometric functions is another difficult topic as students struggle coordinating visual inscriptions that present trigonometric functions—triangle, unit circle, and graph (Presmeg, 2008). After an unsuccessful attempt to create a field of promoted action for *connecting* different inscriptions (Alberto et al., 2019), we prompted students to *reinvent* a sine graph based on the interaction with the unit circle. Supported by continuous feedback, students would establish new sensory-motor coordinations: at first, they would uncover the correspondence between the length of an arc on a unit circle (Figure 6a), and further the correspondence between the sine value on the unit circle and *y*-coordinate on the graph (Figure 6b). Combining these two coordinations (Figure 6c), they draw a line that matches the sine graph, reflect on its construction, and embed their sensory-motor findings into the algebraic notations.

Contrary to the instrumented field of promoted actions where the designers introduce mathematical artifacts, we melted the target artifacts (Shvarts & Alberto, 2021), i.e., deliberately removed a sine graph and parabola from the digital environments. Instead, we created conditions in which students could reinvent those artifacts themselves. Based on our 4-year design research results, the same principle of melting holds for any support the system might provide for the students. For example, if a segment connects the points on a unit circle and grid, the students talk about keeping this segment horizontal and miss the correspondence of sine values across visualizations.

Figure 7. Embodied activities to study histograms. Green arrows represent the degrees of freedom for possible movements and are not visible to the students.



As empirical analysis with eye-tracking shows, cultural perception of histograms requires specific sensory-motor strategies (Boels et al., 2019). We created the fields of promoted actions that would solicit the actions that match cultural sensory-motor strategies for building a histogram. Instead of providing continuous feedback, as in the action-based genre, we constrained the students' actions leaving only one degree of freedom, and presented the motor problem as meaningful word problems.

The students would reinvent histograms to represent the heights of all children in a class by solving motor problems of (1) moving the balls along the x-axis to represent the height of individual children (Figure 7a) and (2) moving the bars along the y-axis to represent how many children fall into a particular interval of height (Figure 7b). Note, similarly to the designs on parabola and trigonometry, the target mathematical artifact—a histogram—was melted (see Boels et al., in press for more details).

Further, the students explore the mean on a histogram by manipulating a balance line (Figure 7c), thus promoting the development of a new form of perception based on their experience of balancing. This design builds on students' natural capacity to estimate weight based on 2D representations, thus resembling the perception-based design (Abrahamson, 2014). However, rendering students' embodied intuitions from other modalities needs further theorization.

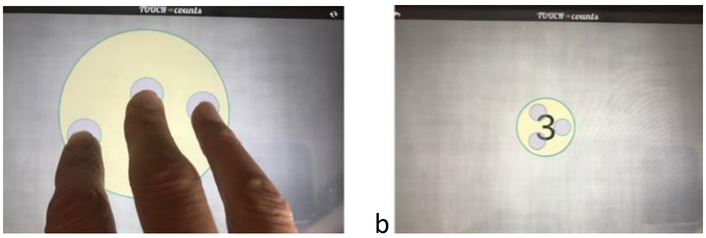
Sensing number sense: the case of Fingu and TouchCounts

The importance of using fingers in the development of number sense has been studied within a growing body of mathematics education literature (e.g., Baccaglini-Frank & Maracci, 2015; Coles & Sinclair, 2018); much of such literature attends to issues of embodiment in learning mathematics, acknowledging that sensorimotor activity such as touching, moving and seeing are essential components of mathematical thinking processes (e.g., Radford 2021). Moreover, many of these studies have focused on children's learning with multi-touch apps.

In particular, Baccaglini-Frank and Maracci (2015) and Baccaglini-Frank et al. (2020) have tried to associate childrens' actions (especially gestures) in certain multi-touch apps with their number sense abilities being elicited. In doing so, we have been pursuing the hypothesis that apps that use multi-touch capabilities may uniquely influence children's mathematical understandings and strategy development concerning number sense, also shared by other researchers (e.g., Tucker & Johnson, 2022). An interesting construct in terms of embodiment in this context concerns conceptually congruent gestures (Tucker & Johnson, 2022), which involve actions with fingers matching the quantity. For example, a group of three fingers placed "all at once" (Figure 8a) creates in TouchCounts a herd of 3 (Figure 8b).

Figure 8: a) Placing three fingers all-at-once on the screen in the Operating World of TouchCounts; b) the herd of 3 that is formed after the fingers are lifted (from Baccaglini-Frank et al., 2020, p. 785)

a b



In both our studies, we use Vergnaud's notion of scheme (2009), focusing especially on operational invariants: the implicit knowledge which structures the whole scheme, driving the identification of the situation and its relevant aspects, and allowing to select suitable goals and inferring the rules for generating appropriate sequences of actions for achieving those goals. While being aware that the notion of a scheme might be seen as being in opposition to embodied cognition approaches, we followed other colleagues. We tried to gain from the two perspectives useful analytical tools to describe crucial cognitive aspects of students' interactions with digital artifacts.

In the former study, we found children's interactions with the app Fingu and LadyBug Count to elicit mostly number sense abilities related to cardinality, that is, abilities elicited in answering the question "How many?" via one-to-one correspondence between physical sets of objects or between or between objects counted and spoken numbers and understanding that the last number spoken in a counting sequence names the quantity for that set. On the other hand, in the latter study, we used the two "worlds" offered in TouchCounts, showing how certain tasks we had designed could elicit fundamental number sense abilities also related to ordinality, which comprises associating number symbols to number words, knowing the number symbols sequence, knowing the number words sequence, and knowing what number comes before or after a certain one (Baccaglini-Frank et al., 2020). The openness of the microworlds in TouchCounts offers the possibility of proposing many interesting and different tasks, as well as "play situations", that can be addressed and solved in countless ways, each with potentially different gestures and schemes.

Experience the dynamics and touch the derivative

The learning environment includes an AR prototype that collects real-time data regarding a dynamic phenomenon (a ball on an inclined plane) during a physical experiment. The sensors collect the data, analyze them, and instantly display their mathematical representations to the students on designated headsets (Fig. 9 right).

The students will thus be able to observe both the real-world experiment and the mathematization of the dynamic object immediately and in real time. When looking through the AR device, the students not only perceive the real experiment, as shown in Figures 9 and 10, but also an elaborated mathematization, which is described below. Figure 9 presents the Hooke's law experiment, which examines the relationship between the mass and elongation of a spring. Figure 10- the Galileo experiment, which examines the relationship between time and distance that a cube travels as it slides down along an inclined plane.

Figure 9. Students collected data by the headset (left side), the display students see, which includes a graph of the mass-length function, order pairs of mass of the cube, and the length of the spring (right side).

The following mathematization-decisions have been made for the development of the AR design. Concerning the Hooke's law experiment, the AR device provides a coordinate system with the weight of the mass on the x-axis and the length of the spring on the y-axis. Whenever a certain weight is added, the elongation of the spring gets larger; the AR device detects the new length and plots it corresponding to the added weight. Adding weight step by step, a diagram is built up showing the relation between the weight and the length of the spring. This diagram overlays the real experiment (Fig. 9, left) and develops with it. A table of values can also be displayed.

For the Galileo experiment, the AR device detects the position of the cube when sliding down a ramp. The AR device provides a scatter plot that plots the time (x-axis) versus the distance (y-axis) at the same time. The distance at a given moment is between the original position of the cube and the current distance along the ramp (see Fig. 10b). The AR device also displays a table of values showing the corresponding time and distance.

These design decisions, accordingly, allow us to enhance real experiments and body involvement with robust mathematical representations like scatter plots and tables of values. For Hooke's law experiment, students can interact with the physical model by carrying the cubes, feeling their weight, hanging the cubes to the spring, observing the elongation of the spring, and interpreting the relation between the weight and the length of the spring by observing the evolving scatter plot. The AR device also shows the length in numbers with a yellow ellipse (Fig. 9, right), which helps the students to better relate the length of the spring to the graph plotted in the diagram.

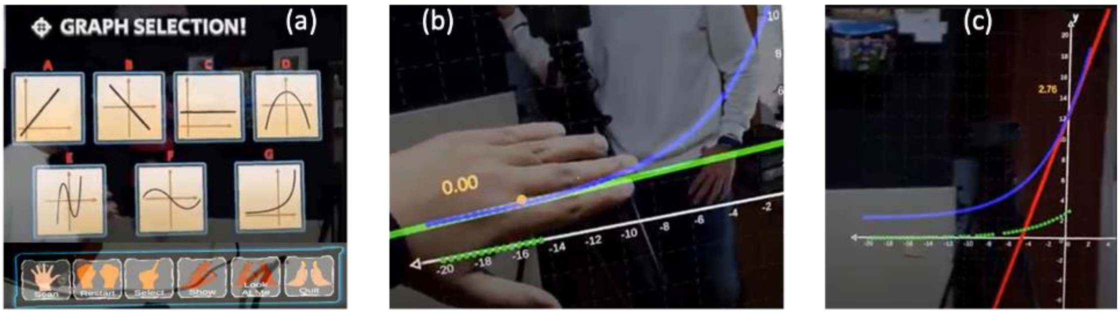
Figure 10. (a) Galileo experiment: a cube travels as it slides down along an inclined plane. (b) The graph and table of values of the distance-time function of the cube movement that one of the students sees through his headset.


a


b

One of the specific aspects of the mathematization in both experiments that contributes to the development of conceptual understanding is the simultaneous presence of different representation modes for the same phenomenon. It is also worth mentioning that the data presented on the headsets are affected by the students' location and the angle the students are looking at the physical objects.

Figure 11. a) The software interface; b) A green tangent line and the derivative graph are created simultaneously with the hand's movement; c) If the student is not close enough to the graph, the tangent line becomes red



An additional embodied learning environment described here is designed with the support of an AR headset called Magic Leap: it is a pair of glasses through which the user can see reality and the augmented objects. In the default interface of the application, the student can choose among seven elementary types of functions (upper part of Figure 11a). In the lower part of the interface, the gestures useful to interact with the technology are recalled jointly with their function (Figure 11a). When selecting one of the functions, a Cartesian system with numbered axes appears, showing the selected function in blue color (Figure 11b). The learner is asked to move his hand along the graph: simultaneously with the hand's movement, a green tangent line appears, and the derivative curve is sketched point by point in the same color (Figure 11b). Moreover, a yellow number denoting the value of the slope of the tangent line is displayed. The student should be close enough to the function graph so that the tangent line is displayed. Otherwise, the line becomes red instead of green, and the derivative graph is not created (Figure 11c).

The design principles underlying this software were inspired by Alberto et al. (2019) work in which students can create the graph of a function by coordinating their hands' movement along a specific constraint. Two a priori hypotheses underlie this learning environment. First, since the environment juxtaposes the function graph, its derivative function graph, and the hand movement, students should come up with conjectures about the mathematical relationship between the two graphs. Second, body involvement should help students find that relationship by 'feeling' the slope behavior as they move and control their hands.

Constructing and diagramming geometry designs

Johnston-Wilder and Mason (2005) pointed out that "A central feature of geometry is learning to 'see', that is, to discern geometrical objects and relationships, and to become aware of relationships as properties that objects may or may not satisfy" (p.4). However, what constitutes learning to 'see'? If learning is "education of perception" (Goldstone et al., 2010), how do we design the learning environments that "take the natural affordances of our long-tuned perceptual systems, which are at their core spatial and dynamic, and retask them for new purposes" (ibid p. 280)?

Echoing Freudenthal (1971), how do we reclaim learning geometry as a living experience for students? Several lines of DBR attempt to answer these questions.

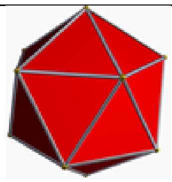
Co-construction of human-scale tangible models

Consider a group of students first building a human-scale model of a geometric body while referring to its two-dimensional diagram and textual description and then using this model to explore the properties of the polyhedron (Figure 12). Several variations of this relatively simple goal-oriented problem-solving design provided various insights into the multifacet nature of students learning to 'see' polyhedra properties (Benally et al., 2022; Palatnik & Abrahamson, 2022; Palatnik, 2022).

Figure 12: The example of collaborative co-construction geometry activity. Left and center—the instruction; Right—participants constructing a model

Using the kit provided to you, your team has to construct a three-dimensional model of the following geometric solid, a polyhedron.

The polyhedron has the following properties: All the faces are congruent; The same number of edges converges at each vertex



The construction problem has several perceptual-material-social characteristics that influence students' actions. When constructing a model, each student has a unique perspective on the task at hand due to the natural constraints of human perception. Moreover, participants must simultaneously consider the dimensions and material of the model (human-size, tangible) and the instruction (hand-held, 2D diagram, and text printed on paper). Constructing the model jointly, students are also constrained by the actions of others and the physical features of the model. To succeed, they must coordinate their actions and make their multimodal referents to the properties of the emerging structure mutually intelligible.

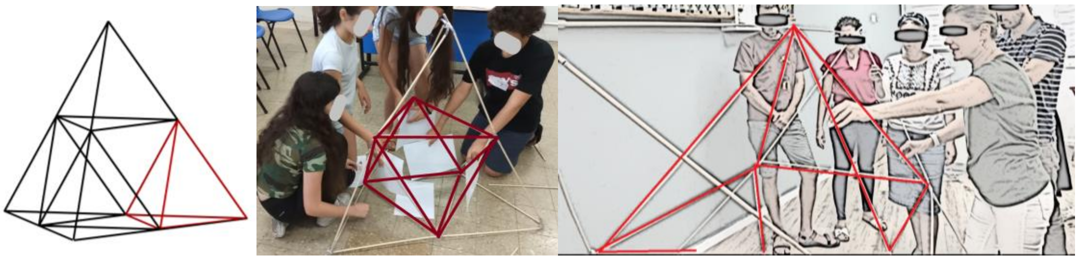
Palatnik and Abrahamson (2022) presented theoretical foundations and empirical arguments for a set of embodied spatial-geometry curricular resources for middle school. In the spirit of Felix Klein's statement, "A model—whether it be executed and looked at, or only vividly presented—is not a means for this geometry, but the thing itself" (Klein 1893, p. 42, cited in Halverscheid, 2019), they conjectured that tasks in which students construct 3D objects are more than "working with manipulatives". In these activities, students use their natural capacities of multimodal perception and collaborative action. We hypothesized that in an attempt to improve and coordinate collaborative actions aimed at building a model, the participants would come to recognize differences in the ways they perceive the model and attune toward each other's perspectives. The need to achieve a common goal will force each participant to explicate reflections on tacit perceptual mechanisms. The student's actions become more complex as the model grows and becomes more composite. In turn, a need to collaborate and communicate these efforts forces participants to engage iteratively in more and more complex forms of reasoning and expression.

Figure 13: An action leads to a shift of attention due to a change of perspective: tacit symmetry becomes visible

Initial designs of the collaborative co-constructive activities were explored in the ongoing DBR project. Palatnik (2022) applied the analytical apparatus of Mason's shifts of attention theory to investigate why and how using physical models of different scales can facilitate learning of (spatial) geometry. The study demonstrated that students' collaborative physical actions and multimodal perception triggered shifts in the focus and structures of attention that, in turn, led to a problem-solving breakthrough. In particular, having tilted the structure onto a vertex, the students perceived an icosahedron as tripartite: two opposing "bases" and a connecting "belt" (Figure 13). Benally et al. (2022) reported that models of different scales landed students different affordances for exploration, for noticing invariant scale-free features of a geometric object and influencing students' collaboration dynamic.

In a similar setting, to solve the problem of comparing the volume of pyramids (Figure 14 left), the students generated the auxiliary problem of defining the shape between four small tetrahedrons (Figure 14 center). Students repurposed sheets of paper lying on the desk as polyhedron faces (c.f. self-invented artifacts in Alberto et al., 2022). They also used their ability as a group to cover all the faces simultaneously. Then, students continued by counting the faces and defining: "The polyhedron between four triangular (pyramids) has eight identical faces. Each face is an equilateral triangle." When a similar activity was conducted during PD, discovering that a simple rotation of the model facilitates seeing its structural features became an insight for teachers: "Now I can not unsee two pyramids with a common square base" (Figure 14 right).

Figure 14: Shaping a void through collaborative action—rendering a contested object into an articulated and inspectable form (red lines are for readers' convenience)

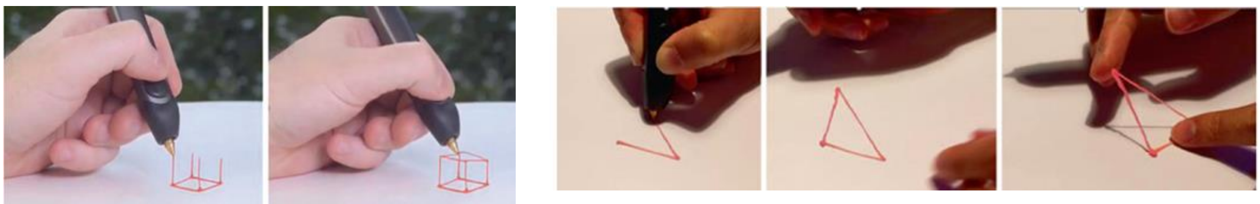


To summarize: co-construction and exploration of tangible models is a robust activity architecture for learning through the surfacing and negotiating learners' perspectives on situated phenomena; students' perception is active and enculturated through participating in the social enactment of the practice of construction; students' critical insights in problem solving can be characterized as shifts in perceptuomotor attention leading to the refinement of geometric argumentation while students' realization of available 3D medium and social setting affordances catalyzes these shifts.

Learning as making: the case of 3D pens

Grounded in Papert's Constructionism (Papert & Harel, 1991), several lines of research (e.g., Ng & Ferrara, 2020; Palatnik, 2023) are premised around the use of 3D printing (specifically, the "3D Pens") as a form of embodied technology for "learning-by-making." 3D printing can extend 2D products to the 3D environment; unlike traditional manipulatives preselected by teachers and which are usually fixed in size and how it is made, 3D printing can provide students with opportunities to generate 3D models flexibly, which Ng and Ye (2022) termed "embodied making" with 3D Pens.

Figure 15. Drawing a cube with a 3D Pen. Creating and manipulating a triangle drawn by a 3D Pen



In this light, the work of Ng and colleagues' research considers the potential transformations that 3D diagramming can induce in mathematics teaching and learning (Ng & Sinclair, 2018; Ng et al., 2018; 2020). Given our interests in embodied mathematics learning, 3D Pens afford increased immediacy and sensory interactions with mathematical representations that are lacking in screen-based tools. This work has shown the affordances of 3D diagramming as a practice of mathematical diagramming in engendering students' geometrical thinking.

The first unique feature of 3D diagramming is the ability to draw in 3D, which overcomes the limitations of paper and pencil and improves the visualization of 3D geometrical objects. For example, one way to draw a cube (Fig. 15 left), is firstly to draw four straight "segments" on a surface to form a square, then four vertical "segments" that join the four vertices of the square, and four more "segments" in the air, while drawing an identical square parallel to the base. It is noted that in the process of drawing such a 3D object, one can visualize vertices, segments, and planes and observe the relations among the 0D, 1D, and 2D objects (Ng & Ferrara, 2020). Besides, 3D diagramming simulates the very process of gesturing; as the hand moves with the 3D Pen, a 3D model is generated. This unique nature of generating 3D models by one's hands affords some interesting hand movement that could not be possible on flat surfaces. Moreover, 3D diagramming supports additional tactile experience by affording students the modality to touch, turn, flip and rotate 2D models drawn. Diagrams that would have been drawn using paper and pencil, such as a triangle, can be recreated and become physical objects that can be held, moved, and turned when drawn by 3D Pens (Fig. 15 right). This suggests the dual nature of embodied making as creating both a diagram and a physical, hands-on manipulative.

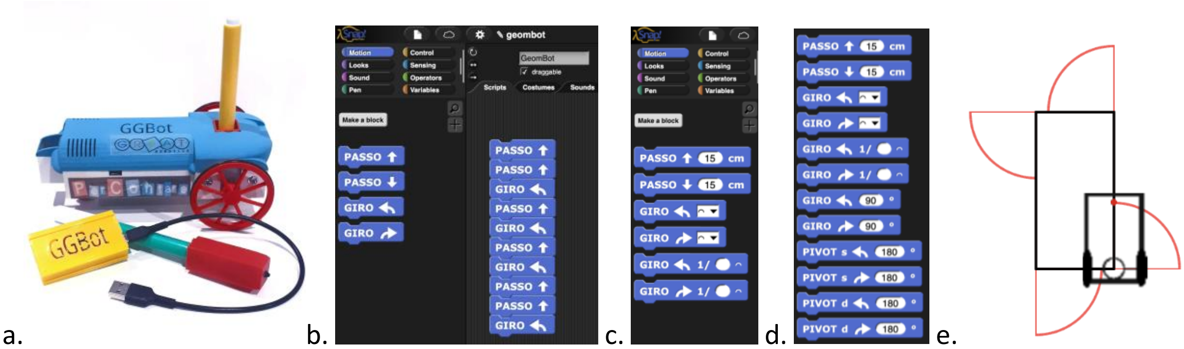
GGBot

The GGBot (short for "GREATGeometryBot") builds on the convergence of physical and digital affordances, combining the well-known strengths and opportunities offered by Papert's original robotic drawing-turtle (more recently developed into robotic toys like the "Bee-bot") and LOGO programming with those of the block-based programming language SNAP! (Baccaglini-Frank et al., 2020). The GGBot can hold a marker between its wheels (Fig. 16a). When the marker is placed and the GGBot executes a code, the marker draws out its path as it moves on a sheet of paper on the floor. A second marker can be placed at the front, on the GGBot's "nose", to leave a trace of its movement when it changes direction. These design features were implemented so that GGBot's traces can provide situated signs (Fig. 16e) that can be elaborated, through appropriate tasks and mathematical discussions, into geometrical notions, such as segment, vertex, angle, rotation, and polygon.

Commands are given to the GGbot through SNAP!, a Scratch based interface that was customarily designed (Fig. 16 b, c, d), and they can be gradually added based on the teacher's needs (Fig. 16c, d). The way in which codes are given to the GGBot is quite different from other robotic toys like the Bee-bot, because the blocks on the screen represent commands (in the machine's language) that can be put together into codes (Fig. 16 b) that can be transmitted to the GGbot via a wifi module. These graphical blocks are virtual objects that "live" on a screen (touch-screen of an interactive whiteboard, tablet, or computer screen); they are concrete enough to be accessible to and shared by the whole class and by each student. Moreover, they constitute another set of artifact signs that contribute to the complex network of signs emerging during activities and that can be put in relation with mathematical signs. Below is an example of the semiotic potential of a figure-to-code task with respect to some of the geometrical notions listed above

(Baccaglini-Frank & Mariotti, 2022). A figure-to-code task consists in giving students the name of a figure and asking them to use the blocks to produce a code, so that when sent, the GGbot draws the nominated figure. For example: “Make a code so that the GGBot draws a square”.

Figure 16: a) GGBot with a marker placed in its posterior holder; b) initial command set and example of code; c, d) commands with parameters for more advanced programming; e) drawing made by the virtual GGBot given the code in b)



Students need to envision the “square shape” as a contour, a path along its border that corresponds to the GGBot’s trace mark as it moves along such a path. Then, such a contour/path must be seen as a sequence of steps, leading to the realization of a code for the GGBot. A spontaneous approach consists in acting: imagining to be the GGBot and walking along the border of the figure (“acting the path”) and associating possible commands to the movements carried out (see Papert’s notion of “body syntonic” learning experiences (1980)). While it is straightforward to identify the four segments constituting the sides of the square and relate them to the step ↑ commands, which are translations, it might be more challenging to decide how to connect these four steps. This requires mastering the complex meaning of the Turn ↱ command (whether it is to the right or to the left), which is a “turn on the spot” (rotation) without translation. In this case, the traces left by the GGBot are thick dots left as the robot changes direction before taking another step forward. A consistent interpretation, leading to completing the drawing, also needs to put these points in relation with one another as the vertices of the square and as centers of the rotations of the external angles of that polygon. So, an essential feature of the semiotic potential of this artifact is its building on the relationship between the GGBot’s global movement and its breaking into steps and turning points and the geometrical meaning of a polygon at a global and analytical level. Moreover, especially if the student “acts the path”, a relationship needs to be conceived between the decomposition and transfer of complex continuous movement (of a child, with many joints working) to only two discrete components - steps and turns - corresponding to blocks of code that will determine the GGBot’s movement and the trace marks left on the paper. The GGBot also has a completely digital version (Fig. 16e) (https://sprintingkiwi.github.io/virtual- geombot-snap), that is more similar to the LOGO turtle (though still in the block-based SNAP! environment)

FUTURE DIRECTIONS

We, the authors of this Research Forum, are excited by the prospects of working together to curate, taxonomize, explicate, counsel, debate, and promote embodiment perspectives on the design and facilitation of mathematics education. Together with our students and collaborators, our collective work ahead falls into six categories: theory, practice, design, dissemination, and academics, as we elaborate below.

Theory

Whereas our epistemological roots vary, we all look to model and foster mathematical learning as a process of mediated negotiation between, on the one hand, biologically endowed sensorimotor capacity for developing perceptuomotor skill and, on the other hand, culturally evolved artifacts, both instrumental and semiotical. These two “hands”—the biological and cultural—are pre-historically “bimanual,” in the sense that evolutionary processes have selected for our species’ natural as well as cultural adaptation—the human ecology is artifactual through and through. Yet, the field of educational research and design is still figuring out the balance between “bottom-up” processes of discovery learning vis-à-vis “top-down” processes of semiotic mediation (cf. Cole & Wertsch, 1996).

What would be the ideal guided reinvention of mathematical concepts? That is, what comes first? Can “top-down” processes give rise to “bottom-up” symbol grounding? We believe that symbolic meaning must be grounded in sensorimotor experience (Harnad, 1990). At the same time, the field might pay more attention to prolepsis (Stone & Wertsch, 1984), a multimodal conversational technique of casting forward into the discursive space a yet-ungrounded structure as a mutually consensual target of sense-making. Micro-analysis of guided mathematical ontogenesis in task-based manipulation suggests that an effective proleptic methodology is to modify a student’s perceptual orientation toward the environment by shifting their attention toward elements in the shared domain of scrutiny that would afford a tighter sensorimotor grip (Shvarts & Abrahamson, 2019, in press). This dyadic “dance” of attention, which serves humans in coordinating joint action, deserves further research. We hope to sustain our dialogue on prolepsis as a research focus for refining our theoretical alignments and contentions. Conducting empirical research studies on this central yet complex phenomenon could enable us to move forward collectively.

Practice

In a longitudinal study, Kosmas and Zaphiris (2023) have documented the instructional gains of introducing embodied learning into classrooms. Liu and Takeuchi (2023) argue for the diversification potential of embodied design specifically for racialized and minoritized students, while Tancredi, Chen, et al.

(2022) apply embodied design to building resources for students with atypical sensorial and motor capacity (see also Lambert et al., 2022), and Shvarts and van Helden (2021) demonstrate the digital reach of embodied design to remote students.

Integrating embodied design into school hinges on adapting and casting these resources as promoting curricular objectives. Yet, successfully interleaving embodied activities into instructional routines stands to change the classroom's epistemic climate, that is, students' implicit sense of what forms of discursive contributions mark legitimate ways of thinking, knowing, and problem-solving (Feucht, 2010). Are students allowed to share their idiosyncratic metaphors (Abrahamson et al., 2012), "be" the graph (Gerofsky, 2011), or invent their own diagrams (diSessa et al., 1991)? As institutional discourse changes around what it means to know a math concept, curriculum experts would need to update assessments, and teacher educators would need to create professional development modules that adapt professional practice.

Design

The brave new "flipped classroom" world is increasingly migrating core instruction from classrooms to homes, only enhanced by the pandemic. Moreover, personal devices (e.g., phones) render digital content universally accessible. Consequently, online learning plays central roles in students' conceptual development. In this context, embodied-activity applications could serve as productive resources, given that they offer dynamically engaging inquiry-based learning experiences.

Multimodality is slowly returning to interaction-based digital educational resources, thanks to engineering breakthroughs in virtual and mechatronic technology. As such, future embodied design may rely more heavily on sensory modalities such as the haptic, tactile, kinesthetic, vestibular, and somatic that by-and-large have been elided from interaction learning due to the technical constraints of early computational platforms. Eventually, as we achieve simulated augmentations of multimodality, we will return digitally to Fröbel's Gift #1, the yarn ball (Abrahamson et al., in press).

Dissemination

Authoring the RF has created a community of practice with much shared common research interests and outreach ambitions. We see value in coordinating enterprises that go beyond individual publications or annual workshops. As a collective, we may stand to mobilize greater resources, garner greater attention, and reach greater audiences. One potentially productive project would be to build a website that curates for the general public resources and information on embodied mathematics learning.

Academics

Change begins at home. As academicians, we need to be the change we want to see at large by creating in our own university departments graduate-level courses, specializations, and even programs dedicated to design-based research on embodied mathematics education. These developments could be supported through establishing special interest groups (SIGs) in annual conferences, such as those run by the Association for Computing Machinery (ACM), the International Society for the Learning Sciences (ISLS), and various regional networks, for example, the American Educational Research Association (AERA). At the same time, our academic activities should remain in dialogue with the wider community of research and practice.

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WORKING GROUPS

SEMINAR

PME-46

MATHEMATICS EDUCATION
FOR GLOBAL SUSTAINABILITY

THE CHALLENGES OF IDENTIFYING RESEARCH "AT THE FRONTIER"

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The metaphor of frontier inspires this WG's topic, for the second year. In the WG organised in PME45, we argued that research considered at the "frontiers" of mathematics education included: a focus on minority/underrepresented populations (e.g., Butler-Barnes, Cheeks, Barnes & Ibrahim, 2021), thus focusing on students who might be considered at the frontier of mainstream schooling; technology and digital environments, especially in the context of the developing world (e.g., Srinivas, Bose & Kumar, 2019), thus considering didactical environments that are at the frontier compared to more traditional ones; and the theories, methodologies, and tools that drive and support such research (Sriraman & English, 2010), thus considering research that lies at the frontier. In the WG realised in PME45, we reflected on this multi-faced understanding of "frontiers" and offered new theoretical and operational ways of dealing with frontiers, from a mathematics teacher perspective. In fact, we built a shared definition for "being at the frontier" in mathematics education, identified current and emerging frontiers ripe for examination, and established a network of researchers interested in doing research in frontiers. The WG proposed for PME46 is the result of one year of interactions, among its organisers, on digging deeper into the construct of frontier, which has been considered a framework for understanding research in mathematics. More specifically, the group of organisers of the PME46 WG met several times to share and discuss the results of searches for articles at the "frontier" of math education in well-acknowledged international journals including *Educational Studies in Mathematics*, *For the Learning of Mathematics*, *Research in Mathematics Education*, *Journal for Research in Mathematics Education*, and the *International Journal of Science and Mathematics Education*. Additional searches were made for papers published outside the "core" of the Mathematics Education community, such as books, which discuss what could be considered items at the "frontier" of math education. We agreed to consider only items published from January 2020 to December 2021.

The goal of the WG for PME46 is to present the results of our joint effort, to provide examples of research that can be clearly considered at the frontier (we propose to consider such research as being at the frontier in a strong sense), and cases that can be considered at the frontier only to a certain extent (i.e., frontier in a weak sense). We will also consider the temporal aspect of being at the frontier, namely that some research topics can be considered at the frontier because little research has been done on that topic, while others would permanently be at the frontier due to their peculiar features.

The first 90-minute slot (Slot 1) will be dedicated to examining examples of weak/strong and temporary/permanent papers at the frontier, discussing possible extensions and inviting for alternative interpretations from the group. Slot 1 unfolds as follows: i) [10 mins] Present the way in which the previous PME45 WG addressed the metaphor of frontiers and how the organisers worked during this year; ii) [15 mins] Introduce six examples of papers being “at the frontier” in a either weak or strong, and temporary or permanent, sense; iii) [30 mins] Small group discussion on the cases shown; iv) [30 mins] Share out with the entire WG, each small groups’ discussions; and v) [5 min] Conclusions.

Slot 2 is dedicated to theoretical approaches that support examination of the frontiers in mathematics education, to formulate research questions, to develop a sound methodology, and to identify what can be considered a result in this line of research. Slot 2 unfolds as follows: i) [10 mins] Summarise the discussions from Slot 1; ii) [10 mins] Recall the six examples presented in Slot 1 with a focus on the methodology used to analyse them; iii) [30 mins] Invite WG participants to analyse a new set of papers, to test the coding method, to develop research questions and to apply the theoretical model which would have been under development; iv) [30 mins] Share with the entire WG and discussion; v) [10 mins] Propose to relaunch/establish a network of researchers willing to continue the work of the WG over the next year.

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CONTRIBUTIONS FROM THE DIDACTIC-MATHEMATICAL KNOWLEDGE AND COMPETENCIES MODEL (DMKC MODEL) TO THE DEVELOPMENT OF THE MATHEMATICS TEACHERS' RESEARCH AGENDA

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During the last 15 years, different authors have created and developed the *Didactic-Mathematical Knowledge and Competencies Model* (DMKC Model) (Godino et al., 2007; Pino-Fan et al., 2017) by articulating several notions of mathematics teachers' knowledge and professional competencies. There is a broad consensus in Mathematics Education that mathematics teachers must achieve a certain level of mathematical competency and must also have specialised knowledge about their professional activity, including that of the teaching and learning processes and the interactions between mathematics and different intervening aspects (such as psychological, sociological, pedagogical, technological aspects, etc.). Due to its importance, the study of this both general and specialised knowledge has been addressed from different theoretical approaches (see Hill et al., 2008; Rowland et al., 2005; Schoenfeld & Kilpatrick, 2008).

PURPOSE OF THE WORKING GROUP (WG)

This WG aims to use the findings from recent research to discuss issues related to the contributions of the DMKC Model, developed by the Onto-Semiotic Approach (OSA), to the development of the mathematics teachers' research agenda. The coordinator is PhD Javier Díez-Palomar in collaboration with researchers from Spain and Panama.

SESSION 1 – THE DIDACTIC-MATHEMATICAL KNOWLEDGE AND COMPETENCIES MODEL

This session aims to introduce the main features of the DMKC Model and focuses on the mathematical dimension of knowledge (*common* and *extended content knowledge*) by using different examples to illustrate it (modelling, hybrid teaching contexts, and teaching of geometry). This session starts with a 10-minutes presentation of the DMKC Model as a didactic approach addressed to a meta-mathematical reflection of mathematics teachers' activities (lesson plans, implemented lessons, etc.). Then, in the following 45 minutes (divided into three slots of 15 minutes each), we will present three examples of using the DMKC Model as a didactic approach to reflect on the knowledge that a mathematics teacher should have to properly manage their student learning: (a) The case of mathematical modelling, looking for potential relationships between the specific principles of modelling and those of the DMKC Model; (b) The analysis of mathematics teachers' knowledge and competencies in hybrid teaching contexts; and (c) The analysis of the mathematical connections and contexts in geometrical tasks designed by prospective primary education teachers using the DMKC principles. In the next 15 minutes, the session will split into three small groups to discuss some guiding questions. The session will end with a 20-minutes plenary discussion based on the conclusions reported by each small group.

SESSION 2 – THE DIDACTIC SUITABILITY CRITERIA

This session aims to delve into the *Didactic Analysis and Intervention Competency*, drawing on the *Didactic Suitability Criteria* (DSC) construct as a tool directly related to this competency. This session starts with a 5-minutes summary of the main contributions from the first session. Then, in the following 15 minutes, we will explain the DSC construct and its six criteria (epistemic, cognitive, interactional, mediational, affective, and ecological). In the next 30 minutes (divided into two slots of 15 minutes each), we will present two examples of using the DSC construct as a tool: (a) To characterise self-regulation practices; and (b) To analyse the practical argumentation when prospective or practising teachers reflect on their own practise (on its design, redesign, and assessment). The final 40 minutes will be focused on a 20-minutes general discussion with the whole group, and a 20-minutes instance for questions and answers from the participants, and a synthesis of the work developed during both sessions.

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AN EMBODIED PERSPECTIVE ON DIVERSITY IN MATHEMATICS EDUCATION

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The topic of diversity and the need to account for various developmental and cultural paths increasingly gains attention within mathematics education. This growing concern in different learning conditions arising from aspects like language, culture, (dis)ability has been reflected in the recent past in their repeated centralization in conference panels (e.g., Prediger on multilingualism at CERME-12; a panel discussion on the consequences of the Covid pandemic for equity in classrooms at ICME-14; Wagner's call for diversification of mathematics education at PME-44). An ESM special issue in progress has been dedicated to the role of racism in math education, following an editorial in response to the Black Lives Matters protests in 2020 (Wagner et al., 2020).

At the same time, embodiment as concerning the role of lived bodily experiences and embodied interactions, including gestures, motor coordinations, eye movements, full body actions, for understanding mathematics has been acknowledged with growing interest within mathematics education (e.g., Abrahamson et al., 2015; Núñez et al., 1999; Shvarts et al., 2021). Following this, it does indeed matter how learners' bodies occupy and act in space we live in (Sinclair & de Freitas, 2019), with our physical and cultural profiles influencing our mathematical thinking and learning. The embodiment lens hence allows for a perspective on diversity that emanates from these conditions as central to knowing and understanding and hence, to mathematical education.

The proposed working group—originally proposed and planned to be held at PME45 in 2022 but had to be cancelled due to circumstances—seeks to extend and widen the exploration of the relationships between embodiment and diversity to understand better the challenges and opportunities of diverse populations and be better able to respond to them. It builds on, synthesizes and extends the work of past PME discussion and working groups on embodiment (e.g., 2012, 2017, 2020), inclusion (e.g., 2018), and marginalization (e.g., 2015) in mathematics thinking and learning.

Main topics, guiding questions and objectives of the WG

We propose two main topics that will guide the work in this group: ***The first one*** concerns the *diversity of bodies* and the influence of learners' *sensory-motor profiles* on learning mathematics. Questions of interest are, for example, related to mathematical epistemology, that is, in how far mathematical cognition and grounded mathematical concepts might differ for people with different bodily configurations and sensory profiles (Krause, 2017). Related to this, we might explore how these differences can shape our approaches for designing the variety of bodies and lived experiences. ***The second topic*** concerns what we call the *diversity of voices* (e.g., genders, ethnicities), captured by the notion of *social-cultural profiles*. Here we wonder, beside others, how similar embodied experiences are expressed differently depending on belonging to minority/majority groups and what kind of instructional support might enable learners from various populations to express their experiences in a mathematical conversation. With this, the ***main objectives*** of this working group are (i) to engage the discourse about the role of the body in diversity and disability with respect to mathematics thinking, learning, and instruction, (ii) to raise key questions for future research and praxis, and (iii) to preparing the basis for a colloquium for the next PME conference.

Activities and structure

Session 1: The first session starts with a short introduction of the organizers, the participants, and the objectives of the WG (10 min), followed by a brief kick-off presentation on general ideas of embodiment and diversity (5 min). We will then explore the diversity of the participants' perspective, experiences, and interests in the topic (20 min) to work in small groups on different aspects of diversity and their relationships to embodiment to gather research questions (30 min). The first session will close with a plenary discussion on the results of the small groups' work (25 min).

Session 2: After a first revision of the first session (7 min), we will give short presentations to introduce the ideas of social-cultural and sensory-motor profiles with respect to embodiment in mathematical thinking and learning (20 min). This is followed by a video-based group work on the influence of social-cultural profiles and sensory-motor profiles on mathematics teaching and learning (33 min). We will then wrap up by discussing and summarizing key topics and questions evolved (10 min) and conclude on next steps and potential future collaborations (10 min).

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COMPREHENSION OF MATHEMATICAL TEXTS: TASKS AND LEARNING PROCESSES

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Progressive education pedagogies often strive to engage young students in authentic activities like those regularly undertaken by expert practitioners. Mathematics education is no exception – some kinds of mathematical activities at the secondary level try to emulate expert mathematicians' practices and, among them, proofs and proving (e.g., Reid et al., 2022). In this working group, we suggest that tasks revolving around the comprehension of mathematical texts (CMT) are authentic activities frequently practiced by mathematicians but very rarely implemented in secondary mathematics classrooms. As mathematical texts frequently include proofs of some kind, reading mathematical texts is often referred to in the literature as *proof comprehension* (e.g. Yang & Lin, 2008; Mejia-Ramos et al., 2012). Mejia-Ramos and Inglis (2009) found that in a sample of 131 articles related to proof and argumentation in mathematics, only three articles focused on students' comprehension of texts containing proofs. Thus, these authors call for more research on reading such texts.

In her doctorate thesis, Elbaum-Cohen (2016) showed how reading mathematical texts consisting of mathematical argumentation provided secondary school students with rich learning opportunities. Selected mathematical texts present problems that students may not be able to solve independently, but they can follow a solution developed by someone else, the text's author. Elbaum-Cohen (2016) demonstrated how through participation in CMT courses, students were confronted with new-to-them mathematics around topics known to them, learned to monitor their sense of understanding, developed communicational skills, and mastered expert-like text-reading strategies. Moreover, students expressed appreciation of the aesthetics of mathematical texts. Marco et al. (2022) studied student comprehension of a specific type of mathematical text presenting "visual proofs" (by means of diagrams, graphs, and mathematical symbols only, without relying at all on natural language) called Proofs Without Words (PWWs) (e.g. Arcavi, 2003; Nelsen, 1993). Marco et al. (2021, 2022) conceptualized the reading of PWW employing *gap-filling*, a construct borrowed from literary criticism, and they showed how even slight modifications in such texts might considerably impact students' mathematical behaviors. These authors showed how a meticulous iterative redesign of PWW contributed to the evolution of pedagogical practices.

- In the first session of this proposed working group, we plan to acquaint and engage the participants with mathematical texts in small group settings. Based on their reading experiences, participants will be invited to raise and discussed pedagogical and research issues worth pursuing. We anticipate that the following may be some of the issues raised:

- Theoretical perspectives for choosing or adapting promising text materials.
- Different types of student learning occurring during the reading of texts, which are different from those occurring in a traditional classrooms.
- Assessing students' success in text comprehension tasks.
- The teacher's role in the implementation of CMT.
- Challenges for students and difficulties characteristic of CMT.
- Direction for using CMT in teacher education.
- Consideration of implementing CMT in different countries.

During the discussions, we will share some theoretical perspectives, the data we collected, main research findings, and proposed practical implications for teaching.

In the second session, we will also strive to form small interest groups based on these issues collected in the first session to establish potential research agendas and consider ways to realize these agendas through follow-up international collaboration initiatives.

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THE AFFORDANCES OF ADVANCED MATHEMATICS FOR SECONDARY MATHEMATICS TEACHING: COMPARING RESEARCH APPROACHES AND THEORETICAL PERSPECTIVES

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More than a century after Felix Klein drew attention to the "double discontinuity" secondary teachers face in their mathematical preparation during university studies, there is ample evidence that the potential affordances of advanced mathematics for secondary mathematics teaching are not being realized. This working group intends to gather the international research community in order to compare and contrast different research approaches and theoretical frameworks, and explicate their underlying assumptions, objectives, methods, strengths, and limitations, with the aim of developing a coherent agenda for continued international research collaborations.

BACKGROUND AND GOALS

There is ample evidence that around the world, secondary mathematics teachers are still experiencing Klein's "double discontinuity", and that the potential affordances of advanced mathematics (AM) for secondary mathematics teaching (SMT) that are discussed in the literature are not realized (Winsløw & Grønbæk, 2014). In recent years, the international research community has been addressing this pressing issue through a variety of research approaches, for example the design of instructional modules integrating secondary mathematics and AM (Wasserman, & McGuffey, 2021); the design of AM courses for teachers that emphasize connections to SMT (Buchbinder & McCrone, 2020); or co-learning communities of mathematicians and secondary teachers (Pinto & Cooper, 2021). In addition, analyses of affordances of AM for SMT have been drawing on a variety of theoretical perspectives, such as Anthropological Theory of the Didactic, Commognition, Boundary Crossing, Mathematical Knowledge for Teaching Proof, and Pedagogical Mathematical Practices. Given this context, we see it as an important task to compare and contrast different research approaches and theoretical frameworks, and explicate their underlying assumptions, aims, strengths, and limitations. Our goals are:

1. To experience different approaches for researching the affordances of AM for SMT and explicate their aims, assumptions, methods, strengths, and limitations (Session 1).

2. To form and elaborate criteria for categorizing research approaches and theoretical perspectives with regard to the affordances of AM for SMT (Sessions 1 and 2).

3. To develop an agenda for continued international research collaboration around the connections between AM and mathematics teaching (Session 2).

ACTIVITIES AND TIMETABLE

Session 1:

- Introduction of Session 1 aims and an overview of research approaches and theoretical perspectives in relation to the affordances of AM to SMT (20 min.).
- In three groups facilitated by the WG leaders, participants will compare examples of affordances of AM for SMT from the literature and from the participants' own experience and discuss how such affordances can be elicited and studied (25 min.).
- Each group will formulate preliminary criteria for elaborating aims, ways of working, strengths, and limitations of different research approaches and theoretical perspectives with regards to the affordances of AM for SMT. Guiding questions will include: How would you characterize the underlying assumptions of a research approach? What might be the strengths or limitations of using a specific theoretical perspective on the affordances of AM for SMT? How can different research approaches or theoretical perspectives complement one another? (45 min.)
- The three groups will present their preliminary criteria in the plenary. (25 min.).

Session 2:

- Review of the work from Session 1 and introduction of Session 2 aims (10 min.).
- Participants will form new groups according to their interest in a particular criterion that was identified in Session 1. The groups will further elaborate methodological and conceptual issues with regard to research approaches and theoretical perspectives on affordances of AM for SMT, including for example: What counts as evidence of an affordance? What constitutes an affordance? (30 min)
- Groups will share and discuss their criteria in a plenary (25 min.).
- The group will identify research trajectories and discuss next steps towards forming and sustaining an international community and future collaborations (25 min.).

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Winsløw, C., & Grønbæk, N. (2014). Klein's double discontinuity revisited: contemporary challenges for universities preparing teachers to teach calculus. *Recherches en Didactique des Mathématiques*, *34*(1), 59-86.Conceptual overlap in approaches to affect: attitude, emotion, motivation and what else?

CONCEPTUAL OVERLAP IN APPROACHES TO AFFECT: ATTITUDE, EMOTION, MOTIVATION AND WHAT ELSE?

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The goal of this working group is to discuss the conceptual overlap among different approaches to researching affect in mathematics education. The increasing diversity of theories and constructs in research on affect is often seen as a threat to the accumulation of mutually meaningful and useful results thus hindering the insight that can be gained. However, the conceptual overlap can also be seen as a strength in this field that reflects the dynamic nature of research approaches to affect. This working group will discuss the origins, characteristics, and consequences of the overlap in approaches to affect and consider potential solutions that may be mutually beneficial across research traditions.

The increasing number of contributions on affective outcomes such as emotions and motivation observed in the last decade (Schukajlow et al., 2023) goes hand in hand with a diversity of approaches to conceptualizing and studying affect. In some cases, we deal with the so-called jingle-jangle fallacy and in others with a conceptual overlap. The “Jingle” fallacy occurs if the same constructs are labelled differently in professional jargon, and the “jangle” fallacy occurs if different constructs receive the same label. This problem has been pointed out on numerous occasions (Marsh et al., 2019). Aside from terms and jargon, many affective constructs uncovered through different research traditions do demonstrate real conceptual overlap. For example, the enjoyment of solving a mathematical problem is a phenomenon that is familiar to humans across different ages, social groups, and cultures. But how should we characterize this phenomenon: is enjoyment a part of the individual attitude, is it a motivational outcome that reflects an interest in problem-solving, is it an emotion or is it something else? The analysis of the theoretical framework on attitudes (Di Martino & Zan, 2015), interest theory (Krapp, 2005) or control-value theory of achievement emotions (Pekrun, 2006) indicates that at least three of these answers might be right correct at the same time.

One of the approaches to deal with the diversity of approaches and to structure the field of effect is to pay attention in research to the key characteristics of affective outcomes such as valence, temporal stability, situational specificity, and objects of affect and how those key characteristics are treated in theory and method (Schukajlow et al., 2023). Some other approaches are to use complementary methods that can examine common phenomena from different perspectives simultaneously. Regardless, transparency of the aims of the research, its epistemological framing, and methodological assumptions need to take into account conceptual overlap and these relationships among constructs.

The goal of this working group, which will meet for the first time at PME43, is to discuss the following topics:

- the diversity of constructs as they are conceptualized in the field of mathematics affect,
- the reasons for, and consequences of this diversity
- innovative approaches for how to deal with the diversity of constructs in research on affect in the future.

ACTIVITIES

First Session (90 min): what is the conceptual overlap and why it occurs?

- Introduce the goal of the WS and presentation of examples of the jingle-jangle fallacy and the conceptual overlap between affective constructs (20 min).
- Small group discussion: What are overlapping constructs and what are the reasons for the overlap and its consequences (25 min). Followed by plenary summary (20 min).
- Plenary discussion on the conceptual overlap in affect (25 min).

Second Session (90 min): how we should deal with the overlap?

- Summary of results of the first session (10 min).
- Small group discussion: what approaches can be used to increase the clarity of affective constructs (30 min). Followed by plenary summary (20 min).
- Plenary discussion on how to deal with conceptual overlap to support common goals in research on affect in the future (30 min).

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INTERNATIONAL PERSPECTIVES ON PROOF: RECENT RESULTS AND FUTURE DIRECTIONS

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This working group (WG) continues the WG on international perspectives on proof at PME 43 and PME 45 (Reid et al., 2019, 2022). The aim is to foster research on proof and related activities (such as proof construction, validation, comprehension, or conjecturing) by offering a platform for the exchange of international research on proof and related activities, especially also integrating new PME members in this discourse. For this, the WG brings together research on proof and related activities with a special emphasis on international comparison and cultural differences, for example, regarding different norms for the acceptance of proof. The WG strives to connect scholars and initiate international cooperation and research on proof and related activities. The long-term goal is the submission of a research forum emerging from this WG.

Proof is a central theme in mathematics education research, and there is an increasing amount of evidence focusing on different aspects, such as the acceptance of proof, different activities related to proof, or curricula and implementation of proof in education (see e.g., Mejía-Ramos & Inglis, 2009; Stylianides et al., 2017). However, most research is conducted in single national and cultural contexts, and only a few studies have compared different nations or contexts (see e.g., Bergwall, 2021; Lesseig et al., 2019; Miyakawa, 2017). It is thus unclear whether research results on proof can be generalized internationally. Still, there is a growing awareness that proof and related activities are at least partially context- and culture-specific, starting, for example, with what constitutes an acceptable proof or what word is actually used for “proof” in different languages (e.g., *argumentation* vs. *démonstration* in French). While laborious, only comparative studies involving multiple contexts, nations, or languages can provide evidence about the generality of findings. These allow for the analysis of differences and similarities regarding, for example, the *teaching and learning of proof* in areas such as curriculum (including textbooks and other teaching and learning materials), student learning and performance, teaching (including teaching practices, teachers’ knowledge, teacher education, and teacher professional development), and assessment.

At PME 43 and PME 45, multiple subgroups were formed, focusing on different areas of interest, such as the study of secondary textbooks or the secondary-tertiary transition. Additionally, a list of more than 30 interested scholars working in the area of proof and proving has been created. The planned WG continues these efforts by sharing information on the past processes within the WG and introducing new possibilities to cooperate within and across these groups. In particular, a joint research project will be outlined, discussed, and refined, which could be a basis for further research within the group and a basis for an international comparison on proof. Based on an initial whole group discussion, individual groups will work on the joint research project, either adapting it based on the interests of the groups or focusing on language and culture-related issues of the proposal. All participants of the WG are invited to jointly plan and shape the study during the WG sessions and after the conference to gather and evaluate the corresponding data, ultimately leading to a shared publication.

STRUCTURE OF THE SESSIONS

The involvement of participants from different countries is essential to the functioning of the group. Over the two sessions, the following activities are planned:

- Introduction to the working group, its aims, goal, and history
- Proposal of the joint research project
- Discussion, refinement, and adaptation of the proposal based on the interests of participants in the working group; possibly in subgroups regarding specific interests and languages
- Whole group discussion of ways to proceed with the joint research proposal

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POETIC METHODS IN MATHEMATICS EDUCATION

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In this new working group, for which we use the acronym POEME, we will demonstrate two distinct but interrelated poetic methods used within our mathematics education research. We will encourage participants to engage in poetic work with transcripts of interviews and of problem-solving conversations. We invite academic dialogue about the ways that poetic methods can draw attention to patterns of voice.

BACKGROUND

Many areas of mathematics education research are committed to the position that mathematical activity is relational, with the consequence that research artifacts, whether interviews or recordings of learning conversations, may demonstrate a layering of voices of mathematical experiences. Poetic methods offer new ways to draw attention to these patterns of voicing in recorded discourse, foregrounding the voices of participants and inviting connections between the speaker and the listener (Fitzpatrick & Fitzpatrick, 2020). This new working group (POEME) is an effort to increase methodological capacity around mathematical poetics, and to establish long-term academic collaborations on theorizing issues of voice in mathematics education research through poetic methods.

The POEME Working Group will engage participants in two distinct but interrelated poetic methods, one focusing on interviews and the other focusing on learning dialogues. First, the group will analyse interviews using a novel rubric adapted from the Listening Guide (Gilligan et al., 2006), a voice-centred, relational method that uses pronoun poems to consider the co-existing voices that occur within a person’s narrative (Helme 2021; 2022). Pronoun poems account for changes over time in relational identities between a student labelled as mathematically low-attaining and her teacher. Second, the group will practice a poetic method (Staats, 2021) that identifies pairs of linguistically similar comments within a problem-solving dialogue that translanguages between English and Somali. Here, poetic methods highlight shifts in shared mathematical foci across many minutes of a learning dialogue.

Goals of the working group

Participants in this working group will improve their facility using at least one method of poetic inquiry, considering the following questions: 1) What dimensions of mathematical experience can be foregrounded by poetic methods in interviews or in learning conversations? 2) What research issues could be investigated through poetic inquiry methods? 3) Would participants like to work towards a collective output, such as an edited volume or special journal issue?

ACTIVITES AND TIMETABLE

Participants are encouraged to bring a 1 to 2 page passage from their own collection of transcribed mathematical interactions for small group practice with poetic methods during session 2. Participants are asked to attend to their ethical agreements as this material may be viewed by others during session 2. The presenters will provide additional transcripts for participants who are not able to bring their own material.

Session 1

- Framing poetics as a domain of voice; goals of working group (15 mins)
- Poetic inquiry method 1 and 2: Discuss methods and sample analyses (Method 1, 30 mins; Method 2, 30 mins)
- Questions, discussion and planning for session 2 (15 mins)

Session 2

- Review of Session 1; organization of groups around transcriptions (15 mins)
- Groups construct a poetic artifact from their transcription sample, with guidance from presenters (30 mins)
- Group presentations (20 mins)
- Discuss questions above; consider collective outcomes (25 mins)

ACKNOWLEDGEMENTS

The presenters thank Professor Claire Halpert, Fardus Ahmed and Emily Posson for their contributions to the development of Somali-English transcripts.

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COLLOQUIUM

UNRAVELING STRATEGIES AND (MIS)INTERPRETATIONS OF STATISTICAL GRAPHS – IN SEARCH OF THE POTENTIAL OF EYE-TRACKING DATA

Organizer: Wim Van Dooren — Discussant: Stefan Ufer

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Statistical literacy is an important competence of nowadays' education. A large part of it refers to handling graphical representations of data. Being able to use a variety of graphs may be beneficial, given that different representations may reveal or stress different aspects of the data. The starting point is understanding graphs of the distribution of a single variable. This implies looking at features of the whole dataset, such as the center, spread, or skewness of a distribution, and thus taking a global rather than a local view of data.

The distribution of a single variable can be represented graphically in many ways, including histograms, dot plots and box plots. Each graphical representation requires a specific interpretation, and has potential affordances and constraints. Recently (e.g., Lem et al., 2013), it has been shown that a range of different strategies and certain misinterpretations may be at play when interpreting these graphs.

The colloquium looks into recent research on the strategies and (mis)interpretations involved in graphs representing the distribution of a variable. The potential of eye-tracking in revealing students' strategies in mathematics has been acknowledged, but research on graph interpretation is scarce (e.g., Strohmaier et al., 2020). The colloquium will show the range of methodological approaches and challenges when conducting eye-tracking studies in this domain, including item design, ways to infer strategies based on gaze patterns or other indicators, and the need for triangulation, for instance by eye-tracking stimulated recall interviews. The overarching challenge seems to be to infer from behavioral data like eye movements the underlying cognition of learners. The contributions do not only discuss implications for further research, but also for education: A better understanding of the range of strategies and possible misinterpretations, but also the possible use of gaze data in instruction itself.

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Overview

REVEALING COGNITIVE PROCESSES WHEN COMPARING BOX PLOTS USING EYE-TRACKING DATA—A PILOT STUDY

Martin Abt, Frank Reinhold, & Wim Van Dooren

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Comparing data sets based on box plots is a challenging task. A typical error is due to a bias caused by thinking the box area is related proportionally to part of the sample, while it is inversely related to the density of the sample. How an area bias exactly affects different individuals, and what strategies individuals who are not affected by an area bias use, is not yet completely understood. We model different cognitive processes to make predictions for solution patterns in six item types, assign students to our a-priori defined patterns, and show that it is possible to validate our hypotheses for their underlying cognitive processes by analysing eye-movement gaze patterns.

This paper can be found in the proceedings, Volume 2, pages 11-18

SECONDARY SCHOOL STUDENTS INTERPRETING AND COMPARING DOTPLOTS: AN EYE-TRACKING STUDY

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Dotplots can increase students' reasoning about variability and distribution in statistics education but literature shows mixed results. To better understand students' strategies when interpreting non-stacked dotplots, we examine how and how well upper secondary school students estimate and compare means of dotplots. We used two item types: single dotplots requiring estimation of the mean and double ones requiring comparison of means. Gaze data of students solving six items were triangulated with data from stimulated recall. Most students correctly estimated means from single dotplots; results for comparison were mixed. A possible implication is that single, non-stacked dotplots can be seen as a step towards teaching students to interpret univariate graphs but further research is needed for comparing graphs.

This paper can be found in the proceedings, Volume 2, pages 123-130

STATISTICAL THINKING AND VIEWING PATTERNS WHEN COMPARING DATA DISTRIBUTIONS: AN EYE-TRACKING STUDY WITH 6TH AND 8TH GRADERS

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Many students tend to perceive a data distribution as a collection of individual values rather than as a conceptual entity (local vs. global view of data). These difficulties seem to persist even after instruction in statistics. This study uses a methodological triangulation of eye-tracking and stimulated recall interviews to examine and contrast 6th and 8th grade students' (N = 49) viewing patterns and statistical thinking when comparing data distributions. Results showed no significant differences between 6th and 8th graders. Regardless of students' grade level, the empirical data confirmed our theoretically derived hypotheses for differences in certain eye-tracking measures (fixation count, saccade amplitude, saccade direction) between students with a local and global view of data.

This paper can be found in the proceedings, Volume *4*, pages *179-186*

SEMINAR

WRITING PME RESEARCH REPORTS: A SEMINAR FOR EARLY-CAREER RESEARCHERS

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INTRODUCTION AND THE GOAL OF THE SEMINAR

PME has provided seminars for the professional development of PME participants on different topics related to scientific activities. The themes of the previous seminars include reviewing PME Research Reports (Gómez & Dreher, 2018) and writing and publishing journal articles (Bakker & Van Dooren, 2019). In this seminar, we extend these PME commitments by organising a program about writing PME Research Reports (RRs).

RRs have two types of papers, namely reports of empirical studies and theoretical and philosophical essays, and this seminar focuses on the first type. The seminar is intended for early-career researchers who are considering writing their first RR proposals for future conferences or have written one or two RR proposals.

The goal of the seminar is to provide some insights into how to write and publish RRs. This contribution type has played several roles in the development of individual research projects. For example, researchers have shared the preliminary findings of their ongoing studies and used the feedback from the audience in the conference presentations to expand their research and later publish their work as full journal articles. This also applies to PhD students who present their intermediate findings and build on the audience's feedback to complete their theses. Published RRs themselves are seen as high-quality publications in the mathematics education community. To take full advantage of these opportunities by publishing RRs, contributors need to be familiar with the characteristics of this contribution type. Submitted papers need not be the report of completed research, but papers are reviewed by researchers with multiple RR publications according to several criteria, such as rationale and research question, theoretical framework and related literature, methodology, and results. The participants of this seminar will obtain helpful knowledge for their future RR writing.

ACTIVITIES OF THE SEMINAR

The seminar consists of three parts described below over 90 minutes in total. We conduct the three parts in the first session and repeat the same program in the second session.

1. Presentation from an experienced researcher. One of the most prolific and popular parts of the PME Scientific Program is RR presentations. This part of the seminar looks closely at what constitutes a good RR paper and offers guidance for early researchers on how to structure their writing to fit the expectations of what constitutes a PME RR (cf. Liljedahl, 2019).
2. Group discussion. Participants are split into small groups where they share the challenges they experienced, or anticipate experiencing, in writing RRs. If experienced participants join this seminar, they are expected to serve as group mentors, moderating group discussions and sharing their experiences. General comments and questions from the group discussions will follow.
3. Communication of pre-submission support (https://www.igpme.org/annual-conference/pre-submission-support/). PME offers pre-submission support for RRs and Oral Communications for novice or inexperienced researchers who are in certain situations and would like to receive guidance from more experienced researchers. We provide information about this system.

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ORAL COMMUNICATIONS

INVESTIGATING THE CONSISTENCY BETWEEN STUDENTS' CONCEPTION OF PLACE VALUE AND A VIRTUAL MANIPULATIVE SUPPORTING (UN-)BUNDLING

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Developing a sustainable understanding of numbers is fundamental to children's early learning of mathematics. Two of the principles forming the basis of our number system are the principle of place value and the principle of bundling and unbundling (Kortenkamp & Ladel, 2014). Virtual manipulatives, such as a digital place value chart, can combine these two principles in new and presumably meaningful ways.

Our project aims to examine whether the automatic bundling and unbundling function of a token-based virtual place value chart is consistent with students' conception of place value and their mental representation of the number system. As part of a larger data collection, we interviewed 13 students in grades 3 and 4 from different schools in Germany. Prior to the interviews, the students took part in a 30-minute intervention, where they completed a number of tasks using a place value chart app. The intervention was framed by a pre- and a post-test assessing place value understanding and regrouping in the number range up to 1000.

Preliminary results suggest that the short intervention with the app did not significantly improve task performance from pre- to post-test. However, an analysis of the interviews yielded various examples of meaningful app interaction: The children used the automatic (un-)bundling function of the app autonomously to visualize addition and subtraction. Furthermore, they provided explanations regarding the functioning of the app that were consistent with the principles of place value and of bundling and unbundling. We also found that some children performed better in the interviews, with access to the app, than in the pen-and-paper tests. Others, however, had difficulties accepting that the app's bundling function proved some of their solutions to arithmetic tasks wrong. They argued incoherently, trying to give reasons for unexpected outcomes, thereby contradicting their initial – and correct – understanding of the app's value preservation.

Based on the interviews, we hypothesize that the app's behaviour is consistent with the students' mental representation of our number system. However, our findings suggest that too limited amount of time spent with the app may not suffice to resolve ingrained misconceptions. In the presentation, further results will be discussed in detail.

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CONSIDERATIONS FOR THE STUDY OF TEACHER'S BELIEFS FROM THEIR ACTIONS

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As beliefs must be inferred by teachers' actions, researchers must take theoretical and methodological decisions about which elements of teachers' actions they are going to focus their attention on, and how are those elements related to teachers' beliefs.

Two notions have been used to state a relation between teachers' actions and teachers' beliefs: *goals* as proposed by Schoenfeld's framework (2008) and *social* and *sociomathematical norms* from Cobb & Yackel's framework (1996).

We state that the complementary use of these two notions provide a finer grain analysis of teachers' beliefs and that they both may be analysed by the Normative Dimension given by the Ontosemiotic Approach (OSA) (Assis et al., 2012).

Teachers' goals relate and can be studied by teachers' configuration proposed by OSA, which constitute the norms that teachers are negotiating with their student to motivate them and to assign, regulate and assess their learning. Furthermore, social and sociomathematical norms relate and can be studied by Metanorms, as proposed by OSA, that is, as norms that regulate norms, or norms that remain unchanged for a certain period.

By studying the norms promoted by teachers, researchers have a starting point to analyze the reasons that led to them. While identifying metanorms allows them to access possible explanations of those norms. Both cases allow the inference of teachers' beliefs from their actions.

In this way, taking beliefs as psychological correlates of the classroom social norms (Cobb & Yackel, 1996) that shape the selection of goals (Schoenfeld, 2008), we state that the use of norms and metanorms, as proposed in the normative dimension given by OSA, is a helpful way of inferring teachers' beliefs from their actions.

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ENGAGING STUDENTS IN MATHEMATICAL MODELING THROUGH DYNAMIC LINKING OF VARIABLES IN M2STUDIO

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Mathematical modeling is the process of translating between the real world and mathematics in both directions (Blum & Borromeo Ferri., 2009), and technology can support students in the process. Dynamic and Interactive Mathematics Learning Environments (DIMLEs) are tools (e.g., Cabri, GeoGebra) that support students' math learning (Martinovic & Karadağ, 2012). Feedback is an important feature of DIMLEs (Leung & Baccaglini-Frank, 2017).

M2Studio (M2S), a family of DIMLEs, is a novel technology being developed to support students' mathematical modeling. Among other things, M2S enables users to create variables, represent relationships among variables, and obtain real-time feedback. For this study, we asked: How does feedback from the dynamic linking of variables in M2Studio support students' mathematical modeling?

A qualitative study was conducted with two 9th graders; one in Geometry class and one in Pre-Algebra. One of the authors conducted individual lab sessions via Zoom in which participants used M2S to work on an open-ended modeling task. Participants' work and use of M2S was recorded and analyzed qualitatively. Our findings revealed that the feedback provided through the dynamic linking of variables in the diagram tile supported students in performing unit analysis, identifying relationships among important variables in the task context, and making calculational connections between the variables. In this sense, M2S acted as a cognitive partner, allowing participants to evaluate and adjust their modeling processes in real-time. The results of this study underscore the relevance of feedback through the dynamic linking of variables in supporting students' mathematical modeling processes and decribes how M2S supports students' work with variables throughout the modeling process. In the presentation, we will discuss further the modeling tasks, methods, and specific details of our results.

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METAPHOR NETWORKS FOR EXPLORING FRACTION CONCEPTIONS

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This communication is to offer a methodology for investigating how students' conceptions of fraction can be understood by depicting their metaphor networks.

According to the conceptual metaphor theory by Lakoff and Johnson (2003), metaphor is a way of seeing the world. Humans interpret their experiences, and conceptualisation is about how individuals perceive their experiences, rather than what the experiences are. In metaphors, the concept that we want to understand is a target domain (in this case, fraction concept), and the source (conceptual) domain is to conceptualise the target concept. Individuals metaphorically map conceptual domains, and the mappings between conceptual domains reflect individuals' views of the concepts. Mowat and Davis (2010) applied network theory to conceptual metaphors and proposed the use of metaphoric networks to investigate individuals' subjective conception structures, with each node representing a conceptual domain linked to other nodes. A key node, highly connected to other nodes, is the hub of the network. As a novel way of examining conception structures, I conducted scripting with sixth-grade Korean students (aged 11-12). Scripting refers to dialogue writing, written by individual students, using imaginary characters that students created. For scripting, I provided a short dialogue as a starting point for writing, called a 'prompt', and students produced scripts about the ensuing situations of the provided prompts in a dialogue format.

I interpreted the metaphoric network of Sophia (pseudonym), who struggled with fractions, by comparing it with the network of Emma (pseudonym) who showed high achievement in mathematics. Compared to Emma's network, Sophia's network has fewer nodes and links, and the nodes are loosely associated with each other. The loose interconnection among nodes implies a lack of conceptual inflexibility. The significance of a hub lies in its repeated use as a source of understanding fractions. The hub of Sophia's network is 'independent number', which refers to the numerator and denominator being considered separately. This conception is linked to the arithmetic errors Sophia showed in fraction operations, such as 5/6+7/8=12/14. Her struggles with fractions could be explained by the compromised hub in a loosely connected network.

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DOUBLE MOVE AS A STRATEGY FOR DEVELOPING LEARNERS' MATHEMATICS DISCOURSE AND UNDERSTANDING

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Examples that teachers choose and use in their classrooms play a vital role in what mathematics is taught and learned, and what opportunities for meaning-making are created in such mathematics classrooms (Essien, 2021; Kullberg, Runesson, Kempe & Marton, 2017). For learning to be meaningful and powerful, teachers need to be aware that everyday ideas and scientific ideas are important and complementary to each other, and hence the double move between both the scientific and the everyday needs to enable meaning making of the mathematical concept for learners (Hedegaard & Chaiklin, 2005).

Using Mortimer and Scott's (2003) framework of meaning-making as a dialogic process in conjunction with variation theory, our study undertook to investigate the role of the teacher in developing learners' mathematics discourse and understanding in a Grade 10 functions classroom. Data was collected through interviews and classroom observations of three teachers in a school. An analytical framework was developed and used to analyse the data collected. The findings that emerged from the study indicated that while the teachers attempted to use everyday examples relatable to learners (wife representing x-value and the husband represents y-value in a marriage situation), the examples were problematic when used during different attempts to 'double move' between the everyday concept of marriage in the South African context and the mathematical concept of functions. Our study highlights the fact that the task of the mathematics teacher in selecting appropriate contexts and moving back and forth between this context and mathematics concept for meaning making is not trivial.

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FEATURES THAT PRE- SERVICE ELEMENTARY SCHOOL MATHEMATICS TEACHERS USE WHEN IMPLEMENTING THE PBL METHOD

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The teachers and pre-service teachers have faced a double challenge in recent years: implementing changes in teaching patterns and adopting pedagogies that they had not experienced while studying. Teaching using the Project-based learning (PBL) method is an example of this. The method presents the pre-service teachers with the challenge of dealing with a change in the teacher's role (Palatnik, 2022).

The purpose of this study was to examine the features of the teaching opportunities that pre-service elementary school mathematics teachers tended to utilize when using (PBL) in their lessons.

The theoretical framework is based on using a ‘pedagogical map’ to present pedagogical opportunities that are exploited by mathematics teachers when teaching with new technology (Pierce & Stacey, 2013). The map was adapted to examine which opportunities are exploited when teaching mathematics using the PBL method by focusing on two aspects: pedagogical-educational and social-emotional.

Three main pedagogical characteristics have been examined: knowledge, skills, and habits. Additionally, social, and emotional aspects have been examined include opportunities that support competency, autonomy, and the need for connection, relatedness, and safety.

Study methodology consisted of case studies. Ten cases in which pre-service mathematics teachers utilized PBL pedagogical opportunities in their lessons were examined. Their aspects were mapped and analysed using a map of pedagogical opportunities for each pre-service teacher.

Findings indicate that by using the PBL method, students became familiar with changing the role of the teacher. The balance between acquisition and construction is altered as well as emphasizing the importance of process work. Furthermore, pre-service teachers viewed PBL as a method supports the sense of competent and autonomous. This study discusses implications for educators and teacher educators in the integration of the PBL method into teaching mathematics. In addition, a tool for mapping pedagogical opportunities using PBL is presented.

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THE PROCESS OF RECONTEXTUALIZATION IN A JOB-EMBEDDED PROFESSIONAL DEVELOPMENT

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Research on teacher learning has largely focused on learning outcomes that gave us little understanding of how the learning process unfolds over time and in connection with teachers' local contexts. Teacher learning has often been studied from the lens of recontextualization as taking up ideas in a primary context and reconstructing them in a secondary context. In contrast to this perspective in recontextualization, van Oers (1998) characterizes recontextualization of learning as continuous sensemaking where teachers wrestle with ideas while determining their goal, examining prior experiences, and "finding out which means are available, investigating which actions make sense to perform in order to achieve the goal chosen, and by relating motive, goal, object, means etc." (p. 481). Within this frame, teachers are not just mere actors in a context, but rather constructors of context. They build on their past experiences and grapple with challenges and tensions between competing goals through the process of recontextualization.

Extensive research highlights the importance of adaptive professional learning opportunities that situated in practice, where teachers make sense through continuous collaboration and inquiry (Kazemi et al., 2021). In our study, we report on the design of a professional development (PD) that supported teachers' ability to make sense of their learning about facilitating classroom discussions through iterative cycles between the PD and their own practice. We co-planned and co-facilitated the PD with eight teachers, three instructional coaches, and three teacher education researchers. Our design allowed teachers to collaboratively reflect on their experiences, jointly identify problems of practice, and revisit their understanding of and use of pedagogical tools and resources.

In our job-embedded PD design, we observed a co-evolution of teachers' sense-making about facilitating classroom discussion during PD and their navigation of problems of practice. We will present evidence of the co-evolution of teachers' sense-making in the PD and in their practice.

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THE COMPARISON OF INFINITE SETS DURING SCHOOL EDUCATION

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The strategies used by students when comparing infinite sets can be categorized into: a) part-whole (i.e. a set which is a proper subset of another set will always have fewer elements than it), b) all infinities are equal, c) 1-1 correspondence, and d) we cannot compare infinite sets (Tsamir, 2001).

This work is part of a larger study of the concept of infinity. In this note, we present a part of the results of this study concerning students' conceptions when they compare infinite sets and how they improved during school studies. For this reason, we asked 132 students in Grade 6, 153 students in Grade 9, and 113 students in Grade 12, whether they agreed or disagreed with the answers of hypothetical students S1, S2, S3 and S4 regarding the comparison of the number of elements of the sets A={1,2,3,4,5,6,7,.....} and B={2,4,6,8,10,12,14,.....}: S1: In set B the numbers are more because they are bigger; S2: We can pair the numbers together so the two sets are the same in the count; S3: Set A has more numbers than the second one because in the second some numbers are missing; S4: The sets have the same number of elements because these are infinite sets. By confronting students with these different responses, we want to explore whether they are influenced by them.

Grade	Agree S1	Agree S2	Agree S3	Agree S4
6	16.94%	33.67%	14.96%	77.31%
9	12.59%	32.52%	13.14%	77.77%
12	3.13%	48.96%	10.52%	77.90%

Table 1: Percentages of students' responses when comparing the two infinite sets

From the above, it follows that more than 2/3 of them in every grade consider that all infinite sets have the same number of elements. However, because the sum of percentages in every line is more than 100, we conclude that many students for every grade agree with more than one answer. The above results are an indication that there is no substantial development in conceptions of infinity during school studies.

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ELEMENTARY SCHOOL TEACHERS' NOTICING OF MATHEMATICAL KNOWLEDGE FOR TEACHING IN THE CONTEXT OF PLANNING, INSTRUCTION, AND REFLECTION

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Previous studies showed that mathematical knowledge for teaching (MKT) is an important contributor to the quality of noticing (e.g., Hino & Funahashi, 2020). In this paper, we would like to describe how a teacher notices MKT and transforms it through noticing cycles of planning, instruction, and reflection. This point of view makes it possible to find a new role of teacher noticing, which is to generate and activate MKT.

Observed was a class teaching multiplication using tape diagrams to second graders. Table 1 shows that Teacher A transformed the purpose of solving Q3 'How many times longer is the length of tape A than tape B', and knowledge transformation occurred through three types of noticing: noticing of students' thinking (external world), noticing of MKT (internal world), and awareness of noticing which is called inner monitor. Thus, consecutive in-the-moment noticing occurred and finally transformed MKT.

Table 1: The Transformation of MKT and Noticing

	Planning		Instruction		Reflection
Situation	Thinking about the purpose of solving Q3	Thinking about the purpose of solving Q3	Thinking about the reasons why the students have difficulty in solving Exercise 1		Thinking about the purpose of solving Q3
MKT Purpose of solving Q3	Verbal expression of times	Verbal expression of times	Necessity of teaching a base amount		A base amount
Noticing			Noticing of students' thinking	Noticing of MKT	Awareness of noticing

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AN INVESTIGATION ON NOVICE MATHEMATICS TEACHERS' RESPONSES TO HIGH POTENTIAL STUDENT THINKING

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The integration of student mathematical thinking (SMT) into classroom learning has been emphasized as a key component of effective mathematics instruction. As a complement to various instructional approaches, teacher responses are crucial to student learning. It is still unclear how to react to and consider students' current ideas emerge during math class discussions.

We are interested in understanding how novice mathematics teachers (NMTs) respond to instances of SMT that have high potential to develop student learning, known as Mathematically Significant Pedagogical Opportunities to Build on Student Thinking [MOSTs] (Leatham et al., 2015). Following a theoretical lens, Teacher Response Coding Scheme (TRC) suggested by Van Zoest et al. (2022), we examined the potential of teacher responses to MOSTs. TRC includes categories that capture: i) Who is publicly asked to consider the SMT (*Actor*), ii) What the actor is doing or being asked to do with respect to the SMT (*Move*), iii) The degree to which the teacher response uses the student action (*Student Action*), iv) The extent to which the student is likely to recognize their idea in the teacher response (*Student Idea*).

We looked across 15 video recordings of whole lessons, each of 40 minutes, which were led by 5 NMTs. It was identified 62 SMTs categorized as MOSTs and determined 97 responses to those SMTs. According to the findings, teachers primarily position themselves as actors and their moves were less productive. Teachers were good at incorporating students' actions and ideas in their response, which means teachers acknowledge students' ideas and honor them through their actions. This indicates that NMTs take SMTs into account but do not know how to use those SMTs to improve students' understanding. We recommend that NMTs be supported by professional development courses on how to react to and use SMTs to enhance students' learning.

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AMPLIFIERS AND FILTERS AFFECTING TEACHER LEARNING OF STUDENT-CENTERED MATHEMATICS INSTRUCTION

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Shifting to student-centered mathematics instruction requires teacher learning, a process in which teachers grow knowledge, refine their practice, and develop new concepts and meaning systems (Horn & Garner, 2022). Scholars identified various factors that amplify or filter what teachers take from a given learning opportunity (Carlson and Daehler, 2019). However, different teachers may draw different – or even contradicting practices or meanings from a given learning opportunity in a PD. Thus, how these factors interrelate within the context of instruction change is undertheorized and requires further study.

This study introduces a system of amplifiers and filters (SAF) comprising beliefs, prior knowledge, level of expertise and organizational support as a framework for analyzing the materialization of learning opportunities. We aim to demonstrate how SAF governs the materialization of learning opportunities into actual teacher learning and instruction change and to substantiate the applicability of our proposed framework. Our data includes video recordings of a 30-hour year-long PD program aimed on student-centered inquiry-based mathematics instruction, observations, and semi-structured interviews with the participating teachers. In the preliminary analysis stage, we mapped teachers' knowledge growth using Carlson and Daehler's model (2019). We then re-coded the data using our proposed SAF framework and created an account for each teacher's learning process, referring to learning opportunities and their materialization.

We present Sara's case, demonstrating how SAF categories interrelated and acted together to amplify her learning and support a successful instructional change. Positive attitude toward uncertainty and exploration of new pedagogies, collegial support and managerial stimulus encouraging students' inquiry, previous experiences and pedagogical knowledge relevant to inquiry-based learning, and an eagerness to learn. All these factors combined and facilitated the materialization of learning opportunities to which Sara was exposed into knowledge growth, concept change, and adoption of new practices and roles.

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A MATHEMATICS TEACHER EDUCATOR'S NOTICING

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Noticing is a skill that is an essential to effectively teach mathematics. Therefore, learning this skill is necessary for teachers. It is also expected that Mathematics Teacher Educators (MTEs) be capable to notice actively in order to teach noticing. Noticing skill of MTEs requires both being able to notice the student's thinking and noticing the teacher's thinking (Amador, 2022). Little is known about MTEs' capability to notice. The aim of this study is to reveal how the noticing process of the MTEs takes place.

The data collection tool of this research, which has a qualitative research design, is the researcher's field notes and reflective diaries. The MTE who conducted this research has over 20 years of experience teaching in both public schools and math education programs. In this study, the MTE spent 48 hours observing 12 PTs as they practiced their teaching in a real classroom setting. The MTE actively used the noticing skill while taking comprehensive field notes to engage with the students' thinking that occurred during PTs' teaching practice and what the PTs did .The MTE's field notes and reflective diaries were analyzed descriptively.

An example of the field notes from MTE: Why is 50% half? PT asked the students. Students could not say anything. The PT did not think that the student would have difficulties at this time. It appears that the student was unable to make connection between the percentage representation and the quantity of fractions. The PT asked the students to switch from percentage to fraction, which caused the student to struggle. At this point, the PT needs to establish the relationship that all these representations show half, even if they appear to be different.

The MTE had the opportunity to develop suggestions for mathematics teaching while capturing students' thinking by establishing the relationship between PTs' teaching practices and students' learning. Taking comprehensive field notes can be used by the teacher trainer to practice the noticing skill.

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LEVERAGING *TOUCHTIMES* AS A TOOL FOR TEACHING

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More easily accessible for young children, touchscreen devices have enabled new opportunities for mathematics teaching in the early grades. Though many educational applications offer high quality experiences with mathematics, skilfully leveraging the affordances enabled by digital technology remains difficult for many teachers. Haspekian (2007) observed that mathematics teachers undergo a process of *double instrumental genesis* when implementing digital technology for pedagogical purposes. This involves dual experiences of personal and professional instrumental geneses where the technological tool becomes a personal, working instrument for mathematical activity *and* a didactical instrument for mathematics teaching. Teachers must understand, predict and plan in advance for their students' processes of instrumental genesis when learning mathematics using the digital technology.

This research examines a case study of a Canadian grade 3–4 teacher's implementation of *TouchTimes* (TT), a specific technological tool, into her mathematics teaching. Interviewed via Zoom in June 2021, Rachel (pseudonym) was a participant in a larger, multi-phase project that was piloting TT. During two 60–80–minute semi-structured interviews, Rachel shared her personal experiences when introduced to TT and its multiplicative models, as well as her classroom implementation of this instructional tool. The interviews were transcribed and analysed. Factors that influenced Rachel's process of double instrumental genesis were identified and categorised. The influence of others emerged as significant. The research questions are: (1) What personal interactions were influential to Rachel's double instrumental genesis? (2) How did these interactions affect the evolution of her professional instrumental genesis of TT?

Rachel's engagement with teachers in the TT project and the success of her students were both influential. She described how the task ideas and the experiences shared by other teachers enabled her to implement the technology. Group discussion about how TT materialises multiplication drew attention to the mathematical affordances of the technology, broadening Rachel's thinking about multiplication and making her more intentional in her mathematics teaching. Rachel's willingness to accept the technology as a viable pedagogical resource increased after witnessing her students' successful engagement with the embodied and relational models of multiplication enabled by TT.

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USING RASCH ANALYSIS TO IMPROVE THE SCORING RUBRIC OF A TRIGONOMETRY ASSESSMENT

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INTRODUCTION

Teachers play an important role in helping students achieve positive outcomes since they play a central role in mediating the content with their learners. It is important for professional development programmes to focus on improving the mathematics knowledge of practising teachers (Bansilal et al., 2015). This study emanated from a professional development programme which was designed to help teachers improve their understanding and teaching of the mathematics that they teach. The focus is on an assessment instrument in trigonometry that was used with 168 in-service mathematics, with the aim of using Rasch analysis to improve the functioning of the items. The research question is:

1. How can an application of Rasch Measurement Model to a trigonometry assessment instrument contribute to an improved scoring rubric?

We focus on 16 trigonometry items from a previous Grade 12 examination paper. When data is analysed, it is common practice to discuss how well the model fits the data. However, with respect to Rasch Measurement Theory (RMT), the requirement is that the data fit the model in order to claim measurement within the models' framework (Andrich & Marias, 2019). The fit statistics are used to help detect when the data does not fit the model as expected and this allows us to diagnose some reasons for the misfit.

Items with disordered thresholds do not contribute consistently to a scale for measuring trigonometry proficiency. Post-hoc rescoring of such items resulted in an improved fit of the instrument. By improving the scoring rubric, the test has been able to distinguish better between items of different difficulty. The ordering of the items according to empirical difficulty also changed after the rescoring, showing that redundant marks can affect the ordering of items in an assessment. The empirical ordering of the items suggested three clusters of difficulty: basic manipulation of trig expressions and calculations; complex manipulations; interpretation of trigonometric graphs.

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USING CONCEPT STUDY TO REORGANISE PRE-SERVICE TEACHERS' MATHEMATICS FOR TEACHING

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According to Davis and Renert (2014), "mathematics for teaching" refers to an understanding of mathematical concepts that enables teachers to teach. It is viewed as both emergent and distributed among teachers. The authors propose the strategy "concept study" as a means of unpacking teachers' mathematics for teaching and thereby allowing them to deepen their understanding of mathematical concepts. Mayar (2019) analysed the strategy with a group of pre-service teachers and illustrated how it enhanced their mathematics for teaching. Rather than thinking it as an enhancement, we see the concept study as an opportunity to reorganise teachers' mathematics for teaching. Further studies on this topic are lacking in the literature.

The study aimed to examine how a group of pre-service teachers reorganise their mathematics for teaching though the concept study approach. Qualitative research was conducted with 20 pre-service teachers in a Brazilian teacher college over the course of five sessions, during which the concept of progression was discussed. Interactions were recorded, and episodes showing changes in participants' communication about the concept were selected.

We identified three ways in which the pre-service teachers reorganised their shared mathematics for teaching: (1) Examining definitions, such as when participants discussed the definition of sequence and progression and specific words; (2) Seeking mathematical connections, for example, when association with linear functions were noted and applications were identified; (3) Exploring the participant's responses during the discussions. Despite having no prior experience as classroom teachers, the concept study approach seemed effective in deepening their mathematics for teaching. We pinpoint "examining", "seeking", and "exploring" as pivotal moments. Teacher educators may therefore benefit from being mindful of promoting these key moments when using concept study. Further research are needed to expand our understanding of the approach in pre-service mathematics teacher education.

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MATHEMATICS TEXTBOOKS: NATURE OF GEOMETRY TASKS AND THE OPPORTUNITY TO LEARN

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Based on changes in the guidelines for Brazilian education and the elaboration of the National Common Core Curriculum (Base Nacional Comum Curricular - BNCC), specifically in the New High School - NHS (2018), changes in textbooks occurred to meet legal demands. Among the changes, integrative projects textbooks (IP) and life projects textbooks (LP) appeared, in addition to the usual Mathematics textbooks, to propose a dialogue between the disciplines. In Brazil, a government program distributes books free of charge to all students in public schools (the last statistical table released in 2020 reports that there were 172,571,931 copies distributed). The program approves a set of works, and teachers choose the one that best suits their school's reality. In 2021, for the NHS, 14 works of IP, 24 of LP, and 14 of Mathematics were approved (each of six books, one addressing Flat and Spatial Euclidean Geometry).

Part of the research developed by the research group TEOREMA – Dialogues between geometry and mathematics education, coordinated by the author, focuses on analyzing mathematics textbooks and has studied the impacts of the NHS on curriculum resources. The results presented here are an excerpt that aims to analyze the characteristic nature of the proposed geometric activities, their role in the scenario of construction and/or application of concepts (Skovsmose, 2011), and the opportunities to learn, especially from tasks (Wijaya et al., 2015).

In total, 52 books (PI, PV, and Geometry) were analyzed, and the results show that the activities of the Geometry books are still based on contexts that contribute little to the formation of the students' critical perspective, focusing on the application of formulas. The best opportunities for critical reflection, elaboration and problem solving, and argumentation are present in the PV and PI books, which address broad aspects of daily life in an integrated manner with other areas, even providing more group experiences.

ADDITIONAL INFORMATION

Research Funding: CNPq – Brazil. Process 421833/2021-1

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USING A COMPARATIVE JUDGEMENT APPROACH TO ASSESS THE PROBLEM-SOLVING SKILLS OF PRIMARY SCHOOL PUPILS

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INTRODUCTION

Problem solving tasks are difficult to assess, and it is tempting to use more closed questions (Blum & Niss, 1991). One possible method, allowing for a range of problem-solving approaches, is comparative judgement. There is increasing interest in the method in mathematics (Marshall, Shaw, Hunter & Jones, 2020) where in contrast to criteria-based methods, comparative judgement involves iteratively comparing pairs of responses, choosing the better response based on the experience of the judge. It has recently been used to look at understanding (Bisson, Gilmore, Inglis & Jones, 2016) and problem solving (Jones & Inglis, 2015).

A large-scale study was carried out to examine the validity of assessing the problem skills of primary school pupils using comparative judgement. An assessment was carried in England in April/May 2022, with 9,133 Year 5 pupils from 203 primary schools, and 2,220 teachers judged on the task. The participating schools also asked their judging teachers to complete an online questionnaire regarding their views of the assessment process. In this presentation, the results obtained from the comparative judgement assessment and the results of the teacher questionnaire are summarised. Based on these results, we consider the validity of the comparative judgement approach for assessing problem solving skills for primary pupils.

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DEVELOPMENT OF AN INSTRUMENT TO MEASURE TEACHER NOTICING FOR INCLUSIVE MATHEMATICS EDUCATION IN ALGEBRA

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Inclusive education is commonly understood as meeting the needs of all students and, thus, poses related requirements for teaching. Hence, mathematics teachers need to possess certain competencies to facilitate the learning of every student (European Agency for Development in Special Needs Education, 2012). This implies the necessity for the development of knowledge for inclusive mathematics education (IME) and situation-specific skills, namely teacher noticing (Kaiser et al., 2015).

In the project Teacher Education and Development Study – Inclusive Mathematics Education (TEDS-IME), we aim to conceptualize, measure and foster teacher noticing skills and knowledge for IME of pre-service and in-service secondary level mathematics teachers. To foster these skills, a professional development programme focussing teaching and learning of algebra was developed combining knowledge from mathematics pedagogical and general pedagogical perspectives. To evaluate the impact of the programme, we constructed a video-based instrument measuring the three facets of teacher noticing – namely perception, interpretation, and decision-making – as well as knowledge tests, building on existing instruments (Kaiser et al., 2015). For the teacher noticing instrument, scripted video-scenes based on typical authentic classroom situations were videotaped. Subsequently, we formulated open-response and Likert-scale items testing all noticing facets and reviewed them thoroughly based on the feedback of 48 experts of mathematics pedagogy, general pedagogy, and special needs education. In pilot studies, analyses using Rasch models demonstrated the potential of the test for a reliable and valid measurement of teacher competencies for IME including all noticing facets and of the progress in teachers' competencies within the implemented professional development programme.

In the talk, the instrument and first results of the ongoing implementation of the professional development programme will be presented.

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LEARNING THE CONCEPT OF DERIVATIVE

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The concept of derivative is central for calculus. Still, many students lack a meaningful understanding of the concept of derivative. This may originate from the acquisition of different aspects of the concept. According to Sfard (1991), concept acquisition is determined by the duality between process and object: concept acquisition starts with learning a concept as process which is finally reified to an object. In case of the concept of derivative, the concept acquisition includes learning three subconcepts *difference quotient*, *differential quotient*, and *derivative function*, each represented as process (with corresponding *operational knowledge*) or as object (with corresponding *structural knowledge*). Both, operational and structural knowledge can appear in two levels, as *concept image (CI)* applicable for visual-graphical representations and as *concept definition (CD)* applicable for a symbolic and a visual-graphical representation (Tall & Vinner, 1981). Using this framework, we designed a unit for learning the concept of derivative in a digital learning environment (20 lessons) along the three subconcepts. Our study aims at understanding more precisely how students acquire the concept of derivative and why many students fail to gain a meaningful understanding.

The sample comprised 113 German tenth grade students. Data was collected by four tests (pretest and one test after the introduction of each subconcept), which were connected by linking items. Scores for all tests were estimated by a unidimensional IRT model. A cluster analysis on students' scores in the four tests revealed two clusters. Students in cluster 1 rather showed structural knowledge for the difference quotient at the CI level in tests 2-4. Students in cluster 2 rather showed operational knowledge at the CD level in test 2 (after learning the difference quotient) and they also achieved this level of operational knowledge for the derivative function in test 4. Both clusters on average did not reach the CD level of structural knowledge for any subconcept. Results suggest that achieving the CD level of operational knowledge when learning the first subconcept (difference quotient) is a necessary condition for reaching this level for the two subsequent subconcepts. Acquiring structural knowledge at the CD level seems hard and to require specific opportunities to learn.

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EXPLORING GLOBAL COMPETENCIES THROUGH A MATH & CP LENS

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Mathematics "is an important tool or 'language' in constructing, explaining, and interpreting the globalized world" (Schell-Straub, 2013, p. 10). Global competencies and development goals are in policy documents, but there is very little research on the integration in k-12 mathematics. Research question: How can mathematics, coupled with coding and digital tools, help students investigate global issues such as environmental sustainability, economic disparities, and equity? The theoretical framework adopted is Kafai and Burke's (2014) Computational Participation (CP) which involves shifting from i) code to actual applications, ii) tools to communities, iii) starting from scratch to remixing, and iv) screens to tangibles. We conducted a cross-case qualitative analysis. There were two research sites, an in-school and an out-of-school site. For the in-school international research site, we observed the participants in their natural setting (naturalistic paradigm). For the out-of-school domestic site, the research team developed the curriculum, for an outreach STEAM camp, through an iterative process (design-test-revise-repeat, Design-Based Research). At the in-school site, students repurposed recyclable tangible materials in an upcycling and shoe design project. They created 3D digital/physical models using geometric and trigonometric concepts. At the out-of-school site, students created a blueprint for a sustainable home with renewable energy in a screen designing software, Tinkercad, exploring concepts such as rotation in the x, y, and z-axis, scaling, and 3D mathematical modelling. Students also coded and programmed tangible circuits (created a prototype of the servo motor to generate wind energy). They thought about angles, parallel and perpendicular lines, and conditional statements.

Implications of the study: This study provides design activities that encourage students to be critical thinkers and collaborative producers who apply interdisciplinary knowledge, including knowledge related to sustainable development practices. The results show how students can be empowered to ask critical questions and engage in rich discussions using in-depth content and context knowledge. There is potential for researchers and educators to explore global issues by adopting the CP lens in mathematics education.

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TEACHERS' CONCEPTIONS OF THE ROLE OF MATHEMATICAL LITERACY TASKS

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There is a consensus among mathematics educators that mathematical literacy is essential for wellbeing in the 21st century (OECD, 2022). Mathematical literacy is often connected to making sense of contextual situations and mathematical work performed to answer a question in context (Julie, 2006). The Math-LIGHT program ("Mathematical Literacy Insights for Growing Horizons") was developed to support teachers' use of mathematical literacy tasks in their classes. The goal of this study is to analyse expert teachers' conceptions about the role of mathematical literacy tasks in the development of students' mathematical and conceptual knowledge.

Teachers' learning communities of practice were created to allow teachers to get acquainted with the principles of the Math-LIGHT program and to encourage its implementation in their classes. The teachers were asked to answer, among other questions, three questions which are under consideration in this chapter: What role does realistic context play in teaching and learning mathematics with Math-LIGHT tasks? Does the context contribute to the development of mathematical understanding? To what extent does context raise students' interest in learning mathematics?

We found that (a) 60% of teachers found that the context evokes students' interest; (b) 46% found that the authentic context helped students connect classroom mathematics to daily life; (c) Only 28% of teachers thought that Math-LIGHT clarifies the essentiality of mathematics to other fields. The conceptual knowledge in Math-LIGHT tasks is often new to the teachers, and they do not feel sufficiently confident to implement the problems in their classes. Our ongoing study is aimed at better understanding the teaching and learning processes linked to the integration of the Math-LIGHT program in mathematical instruction and finding ways to develop teachers' confidence.

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BUILDING MULTIPLICATIVE REASONING USING INTENSIVE QUANTITIES AND SPATIAL REASONING: A STORY OF CLIX MATH

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Researchers have highlighted that reasoning about quantitative relations are central to developing understanding of multiplicative reasoning (see Nunes & Bryant, 2015). Understanding the use of symbolic representations, contextual multiplicative relations, generating correspondences and understanding extensive and intensive quantities are seen as building blocks to multiplicative reasoning (*ibid*). Even though there are multiple trajectories to building multiplicative reasoning, causal linkages between understanding intensive quantities and comprehending and solving ratio and proportion problems are not clearly ascertained. For example, in the problem task, “if 12 laddoos cost Rs 42, then how much would 15 laddoos cost?”, many sixth graders having exposure and involvement in income generating micro enterprises as part of community practices in India, immediately noticed that it was actually the price of 3 laddoos which is to be ascertained and which is a quarter of 12 (Bose, 2015). Such an engagement with intensive quantities is often different from the formal school method of finding the price of one laddoo and raising it to 15. Children frequently encounter many examples of intensive quantities in everyday context such a price per unit (which here was not 1 but 3, the difference between 12 and 15 and also a quarter of 12). The question that this oral communication aims to engage with is to what extent intensive quantities can help build multiplicative reasoning and how digital games can offer affordances for exploring relation between both these measures.

CLIx Mathematics module (https://clix.tiss.edu/) on proportional reasoning has two distinct digital hands-on games among many others on scaling and volume measure offering visual and mathematical affordances for exploring intensive and spatial quantities as linear and area measures and for predicting the variations. Data from students’ hands-on, minds-on engagement with these games will show building of multiplicative reasoning and causal linkages with intensive and spatial quantities.

ACKNOWLEDGEMENTS

Grateful to team members of CLIx Mathematics Team past and present.

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LANGUAGE DEMANDS AND SUPPORTS FOR LINGUISTICALLY DIVERSE (LD) STUDENTS IN INQUIRY-ORIENTED LINEAR ALGEBRA SMALL GROUP DISCUSSIONS

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Collegiate math instructors are increasingly recognizing the overall benefit of inquiry-oriented (IO) instruction. However, IO approaches may not be equitable for students from certain demographics (e.g., gender inequities in Johnson et al., 2020). This analysis explores a question related to equity for LD students: What language demands and supports do LD students experience in IO linear algebra small group discussions?

I adapted Gee & Green's (1998) MASS sociocultural framework to capture language demands and supports along four interrelated aspects of small group discussions: (a) material, (b) activity, (c) semiotic, and (d) sociocultural. The data were one-on-one semi-structured interviews with 5 male linguistically diverse students from one IO linear algebra course taught in English. The 5 students spanned a diverse range of backgrounds (Korean, Vietnamese, Malaysian, Hispanic, and White) and comfort levels with English. The interviews asked each student to share language demands and supports they experienced. Interview responses were analyzed using the constant comparative method, allowing for both a priori and emergent codes.

Below are some of the main language demands and supports that students reported:

- **Material–** *Demand:* Row seating / small desks; *Support:* Face-to-face seating.
- **Activity–** *Demand:* Language-intensive mathematical practices; *Support:* Clarifying instructions.
- **Semiotic–** *Demand:* Verbal dominance and diverse pronunciations; *Support:* Group roles based on math language systems and communication mediums.
- **Sociocultural–** *Demand:* Peer interpersonal communication; *Support:* Intergroup and teacher-facilitated talk, and safer space for communication.

In the presentation, further details will be provided, which suggest the need to facilitate small group discussions in ways that address and leverage these demands and supports.

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THE IMPACT OF EDUCATIONAL OBSERVATION ON THE TEACHER NOTICING OF PRE-SERVICE MATHEMATICS TEACHERS IN MAINLAND CHINA

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Teacher noticing plays an essential role in teachers' professional development. Instead of exploring the status of teachers' noticing, researchers have begun to enhance teacher noticing through various interventions, particularly for pre-service teachers (PSTs) (Lee, 2021). However, little is known about how teaching practices in teacher training programs contribute to teachers' noticing skills. Educational observation (EO) is an essential part of PSTs' teaching practice in mainland china (Zhang & Lei, 2020). This study examined the impact of EO on PSTs' noticing in a Chinese university by using the *Learning to Notice* framework as an intervention tool. A total of 25 PSTs participated in the study. A pre-test and post-test on teacher noticing skills were conducted before and after four EOs at a secondary school. Two teaching clips were used as prompts to assess PSTs noticing levels. The first one focused on teaching strategy (Clip 1) and the second clip focused on teacher-student interaction (Clip 2).

PSTs were required to observe a real mathematics lesson in the classroom during each EO. The data were coded in terms of "What teachers notice" (*What*) and "How teachers notice" (*How*) (Van Es, 2011). The results showed that there were different changes in PSTs' noticing after four EOs. Compared to Clip 1, PSTs' noticing skills increased faster in Clip 2. For Clip 1, in the dimension of *What*, most PSTs (44%) experienced an increase in their noticing level and 12% experienced a decrease. While in the dimension of *How*, more than half of PSTs (52%) experienced an increase in their noticing level and 20% experienced a decrease. For Clip 2, 80% of PSTs showed improvement in *What* and 8% decreased. 72% of PSTs showed improvement in *How*, and only 4% showed a decrease. With the intervention of the noticing framework, the experiences of EOs had impacts on PSTs' noticing levels. An in-depth interview might be useful for unfolding the reasons that PSTs did not change or had a decline in their noticing levels. Suggestions for intervention studies teacher noticing are discussed.

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BOUNDARIES BETWEEN MATHEMATICS AND VISUAL ART TEACHING AND CURRICULUM

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Mathematics (M) and visual art (A) have gained attention for their potential integration in education, providing students with benefits (e.g., problem solving, Brezovnik, 2015). Yet, collaboration between M and A teachers may be hindered by boundaries related to different perspectives or practices, stressing the need for clarifying these boundaries and trying to overcome them (e.g., Akkerman & Bakker, 2011). Various studies use communities of practice and boundary crossing to analyze teacher group work, but studies on interdisciplinary mathematics collaboration at the secondary teaching level are missing. In this paper, we examine the boundaries between MT and AT regarding teaching and curriculum. The study took place in a Greek art school (grades 7-12) over 2 years, initially using ethnography. R1 (first author) visited the school twice a week (field notes, audio-visual recordings, informal discussion with teachers). 17 group meetings (2 math, 5 art teachers, R1) were held every 2 weeks. The first meetings acted as a familiarization phase. Members then co-designed and enacted integrated tasks before the final reflection meeting. Data collected included field notes, audio-visual recordings, teachers' lesson plans, and written reflection reports. We used a grounded theory approach and identified boundaries as epistemological differences that lead to discontinuity in action or interaction regarding teaching and curriculum.

The engagement with mathematical or aesthetic dimensions of objects, and negotiation between theoretical or practical engagement emerges. M appears embedded in AT, whereas A is not essential for M. Also, artists value flexibility in precision, while M can be related to accuracy. Further, A is seen as playful, while M is often considered non-playful. Non-coordination (curriculum content, schedules, classes) also appears. Lastly, M teachers enact individually a restrictive curriculum aiming to students' knowledge, while A teachers use tandem-teaching and enact a flexible curriculum aiming to develop creativity. The study provides a starting point for integrating MT and AT and highlights the need to research how members overcome these boundaries.

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ACKNOWLEDGEMENTS

Funding: “Human Resources Development, Education and Lifelong Learning”, IKY

INSTRUCTIONAL MATERIALS AS A STRATEGIC TOOL FOR MATHEMATICS MIDDLE MANAGER TO STEER INSTRUCTION

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In Singapore, mathematics teachers rely heavily on school-based instructional materials (IMs), which they have thoughtfully crafted for use in their classrooms. These IMs are not simply a compilation of tasks offloaded from curriculum materials. Instead, they often comprised of a set of carefully sequenced mathematics tasks put together, often from various resources, to achieve a certain learning outcome (Chin et al., 2022). In a study, aimed at investigating the evolution of IM use in Singapore, we came across Ken (pseudonym), an experienced mathematics teacher and middle manager for mathematics at Tanglewood Secondary School (pseudonym), who was instrumental in the design and use of IMs in his school for more than 20 years. In this presentation, we present Ken's thinking as an exemplifying case to illustrate what he perceived about the design and use of IMs in his school. We frame the paper around the following research question: How does the design and use of IMs help Ken fulfil his role as a Mathematics Head to enhance mathematics instruction?

Our methodological approach drew on principles of document analysis (Bowen, 2009) and the oral history interview, commonly employed in historical research. Findings were generated from analysis of a semi-structured interview to uncover Ken's thinking behind the design and use of the IMs, substantiated and triangulated from our analysis of three key set of documents—the IMs produced by Ken and his teachers from 2005 to 2020, the textbooks used during the same period, and the corresponding curriculum documents given by the Ministry of Education.

Findings suggest that Ken unpacked the intended curriculum and transformed his understanding into the design of IMs to steer instruction in directions espoused in the syllabus documents. In addition, we highlight Ken's intent of using the designing of IMs as opportunities for developing teachers' pedagogical reasoning to support and change the way his teachers teach. In the presentation, more details will be discussed.

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MATHEMATICAL MODELLING COMPETENCIES OF SECONDARY SCHOOL STUDENTS IN A VIRTUAL LEARNING ENVIRONMENT

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Mathematical Modelling (MM), from a didactic perspective aims to introduce students to the relevance of mathematics and challenge them to apply designated competencies. It begins with a real-world problem that is idealized to be solved mathematically, then it is mathematised to obtain a mathematical model which is investigated till the problem is solved. The results are then interpreted and validated based on reality (Blum & Leiß, 2007; Niss & Blum, 2020). Specific sub-competencies are required during these phases (Cevikbas et al., 2021; Maaß, 2006), yet secondary school students not typically practice such competencies. This study aims to explore students' engagement in MM tasks in the didactical context of a virtual learning environment (VLE). We explore the MM sub-competencies students demonstrate and the VLE features that support them. Participants were 770 ninth-grade students, engaged in MM tasks in a scientific-engineering context which are solved using secondary school mathematics. Research tools include online MM tasks to assess students' sub-competencies during initial mathematising, and observations of students' collaborative investigation of the model and interpretation. Content analysis of students' answers revealed their challenges as well as their progress in mathematising as they gain experience, along with didactic scaffoldings. The observed collaborative learning activities revealed moderate-high levels of investigation and interpretation. Thematic analysis revealed that the VLE supported students' demonstration of sub-competencies via structured learning activities and instructional facilitation of active engagement and collaborative learning. The study contributes to the literature about MM sub-competencies in a VLE.

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ARCHITECTURAL DRAWING AND MATHEMATICAL MODELLING: CONICS, GEOGEBRA AND MORE

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This contribution is part of an ongoing research and concerns the 2D graphic representation of a 3D object. The participants are the students of a course of architectural drawing and survey, in the first year of an architecture bachelor's degree. In such an extra-mathematical educational environment, a mathematical modelling perspective of prescriptive type (Niss, 2015) appears as a tool of investigation and as an educational goal, since students are involved in solving mathematical problems in the design phase of an architectural project, developing a rudimental modelling cycle to go from a real-world object to its mathematization and to the critical interpretation of modelling outcomes. The research questions are: Which are students' recurring misconceptions and difficulties intertwined with mathematical thinking? How to improve students' understanding of drawing as a tool to communicate and visualize objects and their geometrical properties? A joint intervention of teachers of architectural drawing and mathematics took place during regular drawing lessons on conics and surfaces; a GeoGebra applet was proposed to show an ellipse as draggable object to highlight shape variations and invariant properties with respect to movements of the foci. At the end of the first teaching period, students' understanding was tested, asking for a 2D representation of a barrel vault with a semi-elliptical cross section, covering a rectangular base gallery. Two surveys were conducted: a satisfaction one and a questionnaire with dichotomous and multiple-choice questions about synthetic 2D/3D geometry. It seems that students' main difficulties lie in a lack of identification of mathematical objects and in the inability to use their mathematical knowledge in the critical interpretation of their outcomes. This may be since mathematics traditionally is thought at any level as a separate discipline and calls for further investigations. The satisfaction questionnaire shows a positive attitude about the joint intervention: 98% of them think that it is useful to better understand other topics of the course and 93,8% of them find the use of a dynamical geometric software (like GeoGebra) very helpful.

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DIAGNOSTIC TEST AND PROFESSIONAL DEVELOPMENT

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Recent research in the learning of negative numbers has shown that, by the end of primary school, pupils reason about whole numbers. This 'already there' skills of pupils should be considered when introducing negative numbers in middle school (Lamb, Bishop, Philipp, Whitacre & Schappelle, 2018). This arises the question of how to disseminate this knowledge to teachers through professional development (PD) programs. The model of meta-didactic transposition is designed to analyze PD programs focused on knowledge for teaching. In this model, the notion of 'boundary object' is a decisive element: it stimulates exchanges on knowledge for teaching and has to be relevant to teaching practice. The use of a diagnostic test in a PD program focused on negative numbers meets the characteristics of the 'boundary object'. For the knowledge acquired through participation in such a PD program to be truly integrated into teaching practice, it is important that teachers internalize this knowledge: teachers have to become aware of the knowledge acquired and to use this knowledge in their practice. However, there is a lack of studies documenting such an internalization (Robutti, 2018). This is the purpose of the research. The research question is: do teachers who have participated in a PD program based on a diagnostic test, feel that they've acquired knowledge and use it in their teaching practice?

From a methodological point of view, a qualitative analysis of a questionnaire submitted to ten teachers from four schools in French-speaking Belgium who participated in a three-day PD program will be done. The questionnaire consists of five open and five closed questions (Likert scale type). The results highlight the teachers 'feeling of having developed their knowledge for teaching and the ability to express how this acquired knowledge guides their actions in their practice. Beyond the experience presented, this paper invites further reflection on the optimal use, in professional development programs, of diagnostic tests.

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IDENTIFYING CREATIVE PROBLEM-SOLVING STRATEGIES USING EYE TRACKING

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Previous research has shown that creativity and achievements are interconnected, while more creative students perform better on geometry problems and are better equipped with problem-solving strategies. While mathematics education researchers have been trying to deepen our understanding of the problem-solving process, Learning about mental processing during problem-solving using eye trackers is playing an increasingly key role in educational research, particularly in mathematics education (e.g., Andrá et al., 2015). The primary goal of our study was to deepen our understanding of flexibility associated with the production of multiple solution strategies and originality associated with mathematical insight in the problem-solving process.

We conducted an empirical investigation of creativity-directed problem-solving using insight allowing multiple solution geometry tasks (Leikin & Guberman, 2023). We employed individual interviews with a thinking-aloud procedure accompanied by Camtasia software recording and the Pupil Labs GmbH software. The subjects solved tasks while wearing a pupil eye-tracker [Pupil Labs GmbH] and using a DELL 24” touch screen on which the tasks were displayed and on which task the participants could write while tackling the tasks. Camtasia allowed synchronised recording of the thinking-aloud problem-solving process with solution writing on the computer screen, the shifts between different solution strategies, and eye movements.

Eye tracking analysis further reveals differences in thinking processes associated with employing seemingly similar solution strategies. The differences are related to regions of interest, entropy values, saccades numbers, and total fixations. Uniquely our study demonstrated that insight-based solutions have different and identifiable eye-tracking patterns related to more focused regions of interest with smaller numbers of saccades and fixations.

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STUDENTS WITH LEARNING DISABILITIES EXPRESSING MATHEMATICAL CREATIVITY

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One of the ways to strengthen the connections between different mathematics topics and expand existing knowledge is to encourage mathematical creativity (MC). MC is usually assessed by fluency (the number of final solutions), flexibility (the variety of solution strategies or approaches) and originality (uniqueness of the solutions). It is well known that multiple-solution tasks (MSTs) can stimulate divergent thinking and invite students to create a variety of paths leading to the same solution, thus encouraging creativity (Levav-Waynberg & Leikin, 2012). This study focused on students with learning disabilities (LD) and their engagement with MSTs. The research questions are: To what extent do students with LD demonstrate fluency and flexibility when engaging in MSTs?

The study included 12 third grade students with LD from two special education classes located within general, mainstream schools. The students worked in pairs on the following MST “How many arithmetic sentences can you create that have 12 as the solution?”. The researcher carried out mediational interactions based on Feuerstein & Feuerstein (1991). Solutions were coded according for fluency and flexibility.

Fluency findings showed that 96 arithmetic sentences were written, 88% of which included one type of operation. The most common operations were addition (46%) and subtraction (30%). Only 10% of the sentences included multiple operations (indicative of flexible thinking), with three or more numbers (e.g., 5+7–6+6). Students used an average of three operations, created sentences with three numbers or more (also a sign of flexibility), and used a variety of multi-digit numbers (e.g., 21–9, 112–100, 4012–4000).

Results indicate that students with LD can express fluency and flexibility when they engage in MSTs. Providing mediational interactions can play a part in making MSTs accessible and suitable for students with LD.

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WHY LESSONS BASED ON SIMILAR PROBLEMS TAUGHT BY THE SAME TEACHER CAN BE SO DIFFERENT?

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A problem-solving (PS) resource can change significantly along its implementation chain (IC) that begins from PS as intended by the resource designers and continues through PS as planned and enacted by the teacher to PS as experienced by students (Koichu et al., 2022). In some lessons, students struggle with a problem as intended but the designers' intentions for a resource to be a problem are frequently alternated by teachers' and students' intentions and actions. To better understand the implementation of PS resources in mathematics classes, it is essential to identify factors that may affect classroom implementation of the instructional resources intended to be problems for students. In the current study, two lessons based on problems with similar characteristics were studied using the analytical framework of the PS-IC.

Both lessons were based on RBMC (Raising the Bar in Mathematics Classroom) problems (see Koichu et al., 2022) and were taught in 9th-grade classes by the author. Data regarding the intended PS was collected from RBMC teacher guides. Planned and enacted PS were documented using a structured reflective questionnaire. The data on PS as experienced by students was collected using a student questionnaire and from class discussions. The analysis focused on the mathematical and contextual characteristics of each problem along the IC links.

Both problems were in the context of "teachers grade student work", well-familiar to students. The first problem required knowledge of quadratic equations and the second, knowledge of several representations of functions. The analysis revealed that despite many similar characteristics of the lessons, the ICs varied. While in the first case, the experienced PS was very similar to the planned and intended PS, in the second - there was a gap between the intended and planned PS and the enacted and experienced PS.

Comparing the two chains points to three possible explanatory factors: (1) Student mathematical knowledge, (2) Mathematical and social norms, and (3) Personal connectedness to the problem's context. Studying ICs of more problems can help examine these factors and identify other factors that may influence the implementation of PS resources in mathematics lessons.

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STRATEGY USE OF PREESCHOOL CHILDREN ESTIMATING LENGTHS

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Length estimation skills are relevant in many everyday life situations. Estimating lengths means to provide a specific measure without the aid of measuring tools (Bright, 1976). In this process, strategy use becomes relevant (Siegel et al., 1982). There are several approaches that can be distinguished: *Multiple Benchmarks* means to apply a unit multiple times. *Benchmark Comparison* means a mental comparison of another object with the to-be-estimated-object (TBEO). *Eyeball* describes a subjective description of the TBEO such as “because it is so small” (ibid.). In addition, students’ strategy choice may be affected by the TBEO’s size (ibid.).

Empirical studies about the approaches that children use for length estimation predominately focused on children at school (e.g. Siegel et al., 1982). However, length estimation already occurs in the surrounding of preschool children but little is known about (a) their strategy choice and (b) whether it differs for small and large objects. To find empirical indications, four trained student teachers conducted interviews with 182 children immediately before school enrolment in the north of Germany in 2021. The average age of the children is 5.99 years, 41.2% girls and 58.8% boys. One question was to estimate the length of a rope (1.3 m, not small) and another question was to estimate the length of a clamping block (6.2 cm, small). The documented answers of the children are coded deductively (Mayring, 2014) relating to the categories of Siegel et al. (1982). 20% of the data was coded by two independent raters ($\kappa = .863$).

The results indicate that children are already able to estimate lengths using estimation strategies before they start in school (56 % of the estimations were justified with one of the strategy categories, 44 % were not able to communicate their approach). In addition, the strategy choice varies with regard to the two tasks (Rope/clamping block: 52.7%/35% no strategy, 12.1%/3.8% Benchmark Comparison; 17.6%/44.8% Multiple Benchmark; 12.6%/11.4% Eyeball; 4.9%/5% Others). The small object is more often estimated with the aid of estimation strategies than not small objects.

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QUESTIONING ROUTINES IN MATHEMATICS PRESERVICE TEACHERS' DISCOURSE

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Teachers' mathematical pedagogical discourses, essentially their work of teaching during their participation in the classroom discourse, are a key feature of teaching practice. However, scholars who study classroom discourse analysing job of teaching and interactions have failed to provide rigorous discursive lenses to unpack the work of teaching (Mosvold,2016). The aim of this study is to provide discursive lenses of analysing the work of teaching through its focus on the aspect of routine–use. To achieve this, Sfard's (2008) communicational theory has been used due to its great potential to investigate learning through discursive routines. I do this by describing one particular type of work of teaching-questioning routine, and explore the following research question: How can the questioning tasks of teaching be described in discursive terms?

The study reported here involved preservice mathematics teachers participating in a lesson study (LS) that was designed for the study (Gcasamba, 2022). Data was collected through reflective discussions of four preservice teachers and the knowledgeable other (KO) as they participated in LS sessions (lesson planning, enactments of the lesson and lesson reflection discussions).

The results showed that the questioning routines were strongly linked to a process of constructing the routines related to learner activities, presenting a categorization of learners' activities—learner tasks and learner explanations. For example, the results suggested that the questioning routines were used by the teachers to invite learners to participate in the mathematizing activity related to performing calculations and providing mathematical explanations.

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THE VISIBILITY OF MATHEMATICS IN PRINCIPALS' DISCUSSION OF STEM KNOWLEDGE AND PRACTICES

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This Oral Communication highlights the importance of STEM discipline knowledge and skills awareness, especially in mathematics, among school principals. Current research highlights that there is uncertainty surrounding how STEM programs should be enacted, and constituent parts integrated, in schools (e.g., English, 2017). Through analysis of interview data, we outline principals' knowledge of the mathematics in STEM and the implications for STEM education research and practice.

Integrating mathematics effectively into STEM programs faces challenges associated with the experience of teachers and the knowledge base of school leaders (e.g., Geiger et al., 2023). In response to these challenges STEM Capability Sets (SCS) model was developed from research literature (e.g., Geiger et al., 2023) that provide advice in relation to five dimensions: STEM discipline specific and integrated knowledge and practices; contexts; dispositions; tools; and an overarching dimension – a critical orientation to STEM. A design-based research approach (Cobb et al., 2003), involving 88 principals from diverse Australian schools, was used to test the veracity of the SCS. Thematic analysis of interview data, collected at four timepoints, focused on STEM discipline and integrated knowledge of the SCS – particularly mathematics.

Analysis revealed that principals tended to view STEM as a pedagogy rather than an approach to developing discipline knowledge. This may result in discipline knowledge being underemphasised, particularly in mathematics. Principals also tended to rely on staff with specific expertise, and promoting teacher collaboration, rather than providing explicit leadership. There was also a lack of emphasis on appropriate professional learning for staff to develop mathematical knowledge. Further, principals tended to provide resources without a clear understanding of their potential benefits.

The study suggests that Australian school principals have a limited understanding of the role of mathematics in STEM education and need to focus on disciplinary and integrated knowledge and practices to effectively implement STEM education.

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PROSPECTIVE PRESCHOOL TEACHERS' DESCRIPTIONS OF A TRIANGLE: THE CASE OF INDIA

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Recently, India established curriculum guidelines for children ages 3-8 years (NCERT, 2022), which include fostering children's ability to recognize "basic geometric shapes and their observable properties" (p. 61). However, not all observable properties are critical for shape identification. Helping children differentiate between critical and non-critical attributes is the role of the teacher, and the words used to express these ideas can be significant in developing a child's concept image of that shape. This study investigates the way 52 prospective preschool teachers (PPTs) in Goa, India described a triangle. We ask, which critical and non-critical attributes do PPTs mention in their definition of a triangle and what are the terms used to relate to these attributes?

Nearly all participants noted the attribute of three (see Table 1), and the majority referred to *sides*, sometimes calling them *lines*. The term *sides* infers a closed shape while the term *lines*, does not. However, those that referred to lines, added that the lines must be *joined*, or *put together*, indicating their knowledge that the shape must be closed. Few referred to vertices (see Table 1), and of those that did, 79% wrote *corners* instead. The term *corner* is troublesome, as corners may be rounded. Finally, none of the PPTs used the term *polygon*, and less than a third noted that triangles belong to a class of objects called *shapes*, a term used for both 2D and 3D figures and thus not precise. In India, a country with 22 official languages, using precise mathematical terminology is especially important.

Critical attributes	Three	Sides	Vertices	Shape	Closed	Angles	∠ sum 180˚
F (%)	50(96)	46(88)	19(37)	15(29)	14(27)	5(10)	1(2)

Table 1: Frequency of critical attribute mentions (N=52)

Regarding non-critical attributes, 17% added that a triangle has slanting lines and a horizontal line (they used the term *sleeping*), and 17% (not necessarily the same PPTs) stated that the three sides must be equal. One wrote that two sides must be equal. These PPTs are describing the prototypical triangle (Tsamir et al., 2008).

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THE BASIC COGNITIVE CHARACTERISTICS OF ADOLESCENTS WITH DIFFERENCES IN MATHEMATICAL COMPETENCIES

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The pursuit of mathematical abilities includes a broad range of general cognitive skills. Together these skills enable the acquisition, understanding, and performance of various mathematical activities (Kyttala & Lehto, 2008; Paz-Baruch et al., 2022). This study examined the relationship between adolescents' mathematical competencies (MC) and basic cognitive traits, including memory, speed of information processing, attention, and visual perception. We defined students' mathematical competencies as a complex combination of (a) students' achievements relative to the level of mathematics the students studied, (b) students' success in solving mathematical literacy problems (borrowed from PISA mathematical tests), and (c) their mathematical skills as reflected in their scores on the SAT-M test. The sample comprised 84 adolescents divided into two groups: BMC (Basic Mathematical Competencies) and RMC (Regular Mathematical Competencies). Participants were asked to solve basic-cognitive-traits tasks, simple arithmetic exercises, and advanced mathematics tasks (e.g., translation between function representations).

We found that: RMC students were more accurate than BMC students in the simple arithmetic exercises; students' basic cognitive traits tests predicted their success in solving SAT-M and Literacy tests, and advanced tasks; RMC students exhibited significant relationships between the components of MC and advanced tasks whereas such connections did not appear for BMC participants. The study demonstrates that basic cognitive traits do not influence basic and regular levels of mathematical competencies. These findings lead to additional questions about the role of environmental and motivational characteristics in the development of mathematical competencies. We plan to discuss the study and directions for future research.

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COMPARING MATHEMATICIANS' AND MATHEMATICS TEACHERS' PEDAGOGICAL CLAIMS: WHERE IS THE STUDENT?

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Tertiary mathematics teachers (TMTs) and secondary mathematics teachers (SMTs) are key stakeholders in mathematics education. These two communities are responsible for the mathematics education of students that are roughly of the same age. As such, TMTs and SMTs have much to gain from learning from and with one another about issues that are of importance for both communities (e.g., the double discontinuity). However, partnerships between TMTs and SMTs are rare in the field, in part because TMTs and SMTs generally have vastly different perspectives on mathematics and its teaching and learning. There is very little research that can support such partnerships.

This study is part of a long-term research project – M-Cubed – that seeks to address this gap in the literature by studying interactions between TMTs and SMTs within communities of inquiry (Pinto & Cooper, 2022). The design and facilitation of these communities are informed by the literature on boundary crossing (Akkerman & Bakker, 2011) towards encouraging and supporting co-learning between the two sides. The aim of this study is to characterize the social boundary between TMTs and SMTs by comparing pedagogical claims of each side. Data for this study consist of 2 hours of videotaped discussions between five TMTs and five SMTs. Altogether, 151 claims of TMTs and 144 claims of SMTs were analyzed. Each claim was coded according to how it addressed various aspects of instructional situations. Herein, we focus on codes that relate to mathematics students. Analysis shows a substantial difference in whether and how TMTs and SMTs refer to students in their claims. Whereas SMTs tend to situate their claims with respect to individual students or to groups of students, students were generally absent in the TMTs' claims. Moreover, when addressing students, SMTs based their reasoning on different student profiles derived from their teaching experience. In contrast, TMTs generally based their reasoning on their own experiences as students, or on anecdotal cases (e.g., their children). All in all, the place and the role of students in SMTs' and TMTs' pedagogical reasoning emerged as a boundary, and thus as a potential barrier in communication, but also as an opportunity for co-learning.

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INTER-PROBLEM FLEXIBILITY IN WORKING BACKWARDS

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In problem-solving, various circumstances may require switching or modifying strategies. The ability to change them is known as flexibility in problem-solving. Previous relevant research mostly focused on changes in strategies within a problem (e.g., Star et al., 2022). In contrast, we are interested in flexibility between similar problems, in which, however, individual structural elements are varied. For this purpose, we constructed among others a series of three systematically varied problems on working backwards as reversing a process (Assmus & Fritzlar 2022). At one point in the tasks, the difficulty of the transition to the reverse operation was varied by replacing the step "one takes a half" with "one takes a third" or "one takes a third and the other takes two more than the first one". Our aim was to investigate how sixth graders deal with this series, especially with the described "reversal obstacles" and to what extent inter-problem flexibility was shown in the problem-solving processes.

Eleven sixth-grade students worked on the working backwards problems in the context of a videotaped semi-standardized individual interview. The transcribed problem-solving processes were analyzed using qualitative content analysis based on categories using a deductive-inductive approach.

It was possible to reconstruct processes to which no inter-problem flexibility can be attributed and various others in which flexibility was exhibited at different stages of the problem-solving process. The latter include *modifying* and *switching strategies*, whereby switches occurred partly *globally* or partly *locally*. With limitations, the *adaptation of a problem* can also be interpreted as flexible problem-solving action.

The recognition of reversal obstacles, their relevance for action and an appropriate flexible reaction have proven to be by no means undemanding. From this, a goal for problem-solving teaching can also be derived. In particular, in addition to raising awareness of strategies, attention could be paid to metacognitive aspects of monitoring and control, which are important prerequisites for flexibility in problem-solving.

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EXPLORING PRESERVICE MATHEMATICS TEACHERS' COGNITIVE JOURNEY IN UNDERSTANDING CONVERGENCE OF INFINITE SERIES USING DIRECT COMPARISON TEST

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The concept of convergence or divergence of infinite series (IS) is one of the topics in Calculus that most students struggle to understand. Martinez-Planell et al. (2012) found that students have two different constructions of series as an infinite process of adding numbers and series as a sequence of partial sums. As part of an ongoing dissertation project, the current presentation will focus on the preliminary data-gathering activity by the researcher. It aimed to model the cognitive journey of preservice mathematics teachers (PSMTs) as they vertically reorganize their previous constructs into a new mathematical structure while studying the convergence or divergence of IS using the direct comparison test (DCT). Eight PSMTs who had not yet been formally introduced to the topic were given a task to solve, followed by individual interviews. The researcher used a qualitative approach and inductive content analysis based on the RBC-Model for Abstraction in Context (Dreyfus et al., 2015).

Preliminary results showed that the PSMTs had difficulty constructing the notion of convergence using DCT due to their prior constructs in the concept of convergence of an IS. Some PSMTs had a notion that the limit of the partial sums approaches positive infinity when it increases in value, even though the graph of the corresponding partial sums is asymptotic to a horizontal line, or it approaches a finite value. They also had a notion that a sequence of corresponding partial sums should have a defining formula to be considered convergent. Their difficulty in understanding the idea of an IS converging to a limit is due to the subtle shift in thinking about the terms of the series to thinking about the overall sum. Hence, it resulted in the difficulty of using the DCT, even though they had recognized the convergence or divergence of the other given IS used for comparison. The findings of the current study provided basis for reflection by the researcher in conjecturing specific activities that would help students construct a deeper understanding of the convergence or divergence of IS.

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PROMOTING MATHEMATICAL MODELLING COMPETENCIES AMONG LEADING TEACHERS

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Mathematical modelling (MM) is a circular process involving several phases, which enables the connection between the real-world and mathematics (Kaiser, 2017). To spread MM widely through scalable PD, leading teachers should be trained to help teachers integrate modelling activities into their classrooms (Prediger et al., 2019). Their training involves the development of their MM competencies as PD learners, as well as their abilities to face the challenge of leading PD programs as PD leaders.

In this study, we examine whether PD focusing on MM contributes to the modelling competencies of 36 leading teachers who took part in this program. Furthermore, it examines their transition from PD learners to PD leaders. Throughout the program, teachers' competencies were explored by analysing solutions to modelling problems, while following each phase involved in MM. Second, based on reflection and self-reports, we characterized the PD design in terms of effective PD elements (Darling-Hammond et al., 2017) from two perspectives; as PD learners and as PD leaders.

Results indicated that the leading teachers demonstrated MM competencies for each of the MM phases examined. A continuous improvement was observed in their competency to perform the mathematization phase, which is where a transition from real world to mathematics occurs while choosing the appropriate mathematical model. Further, feedback and reflection, as well as expert support, were the most significant elements of their learning process. Based on their experience as PD leaders, the leading teachers reported that content focus and active learning were the most influential elements in their role as PD leaders. This study contributes theoretically to the literature of leading teachers in the context of MM and offers insights into its scalability affordances.

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VISUAL PROOFS FROM HIGH SCHOOL STUDENTS AND TEACHERS' PERSPECTIVES

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The potential of visual proofs to facilitate mathematical insight has been pointed out previously (Arcavi, 2003). Research on the subject accumulates (Marco et al., 2022). However, teachers and students avoid using visual considerations. Harel and Dreyfus (2009) studied whether high school students accept visual proofs as legitimate. Here we explore the teachers' perspective on visual proofs. We ask: 1) To what extent do high school teachers accept visual proofs as legitimate? 2) To which extent and for what purposes do high school teachers prefer using visual proofs over algebraic proofs?

We used a set of four mathematical statements, each with two proofs: a visual and an algebraic one, taken from Harel and Dreyfus' (2009) study. We designed a questionnaire based on eight personal interviews with mathematics teachers. The questionnaire was administered to 121 high school teachers. The findings show that 69% of the teachers accepted visual proofs as legitimate. 50% of the respondents indicated that visual proof is perceived to be more comprehensible than algebraic proof. 72% of the respondents replied that they would give an A score to visual proof as an answer in a math exam. Most teachers (82%) reported they would prefer to use both visual and algebraic proof in their classrooms. Only 7% reported they would not use the visual proof at all. However, 64% of the 74 teachers who accepted visual proofs believed other teachers would not accept them as legitimate as much as they do.

Our findings suggest that despite prevailing positive attitudes towards visual proofs among high-school teachers, there is a common belief that "other teachers would not accept them." These findings resonate with Harel and Dreyfus' (2009) findings about high-school students that accept and prefer visual proofs but do not believe their teacher would accept them. We suggest that more attention should be dedicated to visual proof in mathematics teachers' training and professional development programs.

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ASSESSING MATHEMATICAL CRITICAL THINKING SKILLS – AN ESSAY-TEST USING THE EXAMPLE OF THE MEAT BAN IN SCHOOLS

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Critical thinking can be defined as effortful and consciously controlled thinking which is composed of attitude, knowledge, and thinking skills (Halpern, 2013).

In this paper, to focus on the mathematical part of the thinking skills, we specify *Mathematical Critical Thinking Skills (MCTS)* in three dimensions: The first dimension "Analysis and Problem Solving" includes, for example, the skill integrating information from different sources for solving a partly mathematical problem. The second dimension "Interpretation and Data Literacy" includes, for example, the skill interpreting numerical relationships in tables and diagrams. The third dimension "Argumentation and Communication" includes, for example, the skill using partly mathematical information and data to support an argument.

To capture MCTS empirically, 49 German students completed an essay test. In the main task, students are asked to decide whether or not meat should be banned in German cafeterias and dining halls at schools. This decision should be made based on four individual tasks dealing with health and sustainability aspects of meat consumption. In addition, a detailed reasoning for this decision is demanded. The students' documents were rated based on the MCTS described above. For example, one student calculated several CO_2-emission values and argues (briefly paraphrased):

1 kg beef emits 13.6 kg of CO_2*. 1 kg vegetarian burger patties only emit 1.1 kg of* CO_2*. So not eating meat is climate-conscious.*

This passage gives an indicator that the student has used the skill using of partly mathematical information and data to support his argument. The more of these indicators we find, the better the test grade will be.

The data depict a broad range of MCTS, provide indications of which of these skills are better or worse mastered by students, and suggest how such skills can be fostered in mathematics classrooms.

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MATHEMATICAL CONNECTIONS AND CONTEXTS IN PROSPECTIVE ELEMENTARY TEACHER TRAINING

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According to the NCTM (2000) learning is deeper and more durable if learners can connect mathematical ideas. Establishing good mathematical connections is a powerful tool for learning mathematics. On the other hand, it is known that greater contextualization of mathematics results in more immediate access to knowledge by the students. A professional task is designed and implemented to highlight the value of making connections to foster rich geometric activity and to recognize the role that contexts play in making connections. The study was carried out with 250 Prospective Teachers (PT) from an Elementary Education degree. They are asked to pose geometric school activities that they would propose to primary school students based on different real-life contexts. A first analysis was carried out to identify the connections achieved by the PTs considering the categorization raised by Vanegas and Giménez (2018), subsequently, the type of context is also determined following Martínez (2003). Most of the proposed activities (78.6%) were in evoked contexts, where attention is on the mathematical objects and not in the processes required for solution. In terms of connections, 63.6% of the proposals refer to a metaphorical connection, associated with direct visualizations. It was found that the contexts with measurements familiar to the PTs generate the more powerful connections, for example, in those images that used small-sized organisms such as protozoa, the focus on usual geometric objects and the tendency to generic interdisciplinary connections stand out. But in contexts closer to the PTs, such as the crystals of snow, their proposals try to generate mediating connections and evoke more advanced geometric elements (for example, symmetry).

This study was conducted within the framework of the projects: ANID/PFCHA 72220049 (Chile); PID2021-127104NB-I00 (MICINN, FEDER, UE) and PID2019-104964GB-I00 (MICINN -Spain).

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EXAMINING ACCURACY AND STRATEGY CHOICE ON THE ESTIMATION OF LENGTH

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The accurate estimation of lengths is one important factor that becomes relevant in many situations of daily life. To obtain most accurate results, different strategies can be employed, such as the unit iteration (mental or active iteration of a consistent unit), benchmark comparison (comparing the given length with a memorized benchmark), or decomposition/recomposition (decomposing the object into parts, estimating their lengths and recomposing the estimated values) (e.g., Siegel et al., 1982). Choosing an appropriate strategy can be considered as an indicator for estimation skills and may be related to the estimate's accuracy (Joram et al., 2005). In our study, we analyze how estimation accuracy is related to the use of an estimation strategy.

In this regard, ten students from Germany (8 third and 2 fourth grade) solved 16 length estimation tasks on various objects and explained their approach in a follow-up interview. The estimated objects differed in terms of various properties as proposed by Hoth et al. (2022), while the strategy use was coded deductively with regard to the strategies mentioned above. The results indicate that most of the students used unit iteration or benchmark comparison (34% each of the estimates), and only 17% used decomposition/recomposition. In 15% of the cases, the students did not provide any strategy. Regarding the accuracy of these estimations, missing strategy choice was related to four times as much deviation (222%) compared to students who described their strategy use (64% deviation). Analyzing the 20 most accurate estimations (deviation ≤ 10%) shows that 85% of these estimates were reached by an estimation strategy. In addition, the three most accurate estimators often used decomposition/ recomposition with a benchmark i.e., referring to their finger width of 1 cm or other body parts. These results indicate that there is a need to explicitly discuss estimation strategies in class in order to support students in their flexible estimation skills.

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MATHEMATICS TEACHING STYLES OF TAIWANESE PRE-SERVICE ELEMENTARY SCHOOL TEACHERS: TEXT MINING AS THE METHOD OF ANALYSIS

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The objective of this study was to explore the mathematics teaching styles (MTS) of pre-service elementary school teachers by analysing their teaching demonstrations and self-reports, based on Conti's PALS scale and theory (Conti, 1989). To identify indicators of MTS, we employed text mining as our primary method of analysis.

In this study, we explored MTS of pre-service elementary school teachers based on the sequence of events that occur in the mathematics classroom (as Figure 1). Specifically, we examined MTS from four dimensions: lessons and activities plan (Design, D), instructional strategies (Methods, M), assessment (A), and interaction (I). The survey method was used in this study and data was collected by questionnaires and observation tables. A total of 205 questionnaire responses and their teaching demonstrations were collected and analysed by exploratory factor analysis (EFA), cluster analysis and text mining. To identify important words and phrases associated with each teaching style, we calculated vocabulary weights using text mining techniques. For Chinese word segmentation, we used a Python-based program called "Jieba". We analysed the teaching styles demonstrated by the participants during their teaching demonstrations and the language they used when teaching mathematics.

The results indicate that pre-service elementary school teachers in Taiwan exhibit four different mathematics teaching styles: inquiry-based teaching, motivational triggering, creating a positive learning atmosphere, and student-centered teaching. After analyzing observed teaching demonstrations, four styles were identified: student-centered plus, slightly student-centered, AI teacher-centered, and ADM teacher-centered. By integrating these two aspects, the mathematics teaching styles of pre-service elementary school teachers can be classified into four types: cognitive-student-action-teacher, cognitive-teacher-action-student, cognitive-student-action-student, and cognitive-student-action-student. While text mining proved useful in identifying important words and phrases, it was not sufficient as an indicator. Therefore, future research may consider extending the analysis from key vocabulary to complete sentences for further exploration.

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USING RELAY JOINT TEACHING IN A MATHEMATICS PROFESSIONAL LEARNING COMMUNITY – A CASE STUDY OF ONE ELEMENTARY TEACHER'S QUESTIONING SKILLS

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The paradigm of professional development shift from traditional attending workshops to the professional learning community (PLC). The main idea of PLC is that a group of teachers advance their pedagogical knowledge and pedagogical content knowledge through peer learning, with the purpose of improving their students' learning outcomes (Darling-Hammond, Hyler, & Gardner, 2017; Schumacher, Taylor, & Dougherty2019). The purpose of this study was to investigate the impacts on one elementary teacher's questioning skills using "relay joint teachings" as the main activities in one rural elementary school located at Miaoli County, Taiwan.

There were eight teachers in the PLC group. Data were collected throughout ten semesters. Two teachers would practice relay joint teaching activities each semester t, three sessions for one teacher. Each session last for 3 class periods, including teaching activities, one teacher in this PLC group and the researcher each taught one mathematics class consecutively, and follow-up group discussions regarding both classe. All PLC activities were recorded on video for further analyses.

The main findings were the changes of the questioning skills of the teacher in this case study across this five-year period, which included types of questions (e.g., closed-ended question vs. open-ended questions) and the way these questions were asked. The findings indicated that using relay joint teachings did make changes on inservices teachers' teaching behavior. The results of this study might provide a different method for conducting professional development.

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FOSTERING MATHEMATICAL KNOWLEDGE THROUGH USAGE OF META-COGNITIVE STRATEGIES: AN EXPLORATIVE STUDY

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The study focused on pre-service middle school student teachers' (PMST) sense of the development of conceptual understanding, procedural fluency and constructing knowledge in mathematics learning. The research also emphasized the use of meta-cognitive strategies like think aloud, KWL (What I know, What I want to know and What I learned) charts, self-reflection sheets and journals enabling PMST to think about both the mathematics they teach and the mathematics learning and problem-solving experiences they create for their students. Researchers especially want to get answers for how PMSTs are ready to use meta-cognitive strategies in teaching middle school students, with the hypothesis that there is no significant difference between procedural knowledge building with meta-cognitive strategies and the accustomed way of teaching. The 15 participants of this study are 4th-year student-teachers from Russia who had a pedagogical practice in 2022 at 8th grade. The practice lasted for six weeks. PMST used meta-cognitive strategies in teaching geometry (Construction of quadrilaterals and triangles) to students. While teaching mathematics PMST used the strategies of meta-cognition as an instruction design (Ingole & Pandya, 2017).

In this research, a mixed (quantitative and qualitative) method was used for the collection of data. The interviews were partially structured as the initial questions were prepared, but follow-up questions were posed without a predetermined design. The finding of the study shows there is a significant difference between procedural knowledge-building and meta-cognitive strategies. The quantitative results indicated that university student-teachers have moderate positive perceptions, attitudes and meta-cognitive awareness levels. The qualitative findings of study themes emerge as follows: 1. PMSTs (prospective teachers) use meta-cognitive strategies in their own university studies too before teaching in mathematics pedagogy. PMST gets into the habit of using meta-cognitive strategies for effectively regulating procedural knowledge. 2. PMST gives more importance to how and why to solve mathematics problems instead of only what will be the steps for solutions. Moreover, PMST expressed interest and felt a need to increase their level of expertise when developing their students' meta-cognitive abilities. 3. PMSTs were inclined to believe more in procedural and conditioning knowledge rather than only declarative knowledge which strongly associates to enhance problem-solving abilities.

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A MODE STUDY ON PAPER- AND COMPUTER-BASED MATHEMATICS TESTS IN THE CONTEXT OF TIMSS 2019: THE CASE OF GERMANY

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Digitization has proceeded in Germany during the last decades, which can also be observed in various instances of teaching and learning. In a similar vein, large-scale assessments have recently switched from paper-based to computer-based tests. In TIMSS 2019, this change was accompanied by a mode effect study (Fishbein et al., 2018). Two years before the main survey, 847 fourth-graders participated in this study and were randomly assigned to take either the paper-based or the computer-based achievement test in mathematics and science on the first day, and vice versa on the second. In this paper, we restrict ourselves to the results from the mathematics test and address the question to what extent students' achievements depend on the survey mode, which is considered an important step in validating test score interpretations (AERA et al., 1999). We firstly conduct a generalizability study to assess the amount of variance in test scores that is explained by mode effects. Secondly, we investigate to what extent the relations between difficulty-generating characteristics (e.g., content and cognitive domains, Cotter et al., 2020) of the test items and students' achievement differ by mode. The results show that the assessment of students' mathematics achievement is indeed affected by small mode effects. In the computer-based test, students scored slightly lower (OR = 0.87, p = .006). This particularly holds true for tasks addressing data and problem-solving. In the paper, the relevance of these findings for large-scale studies and educational practice in German mathematics classrooms is discussed.

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INTERPLAY BETWEEN MODES IN MATHEMATICS TEXTBOOKS

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The use of different modes (e.g., words, mathematical symbols, images) is central to encounter different perspectives of the same mathematical idea, and a combination of different modes generally contribute to a deeper understanding, than each mode separately (O'Halloran, 2005). At the same time, research has shown that combinations of modes, e.g., in mathematical textbooks, are challenging for students to interpret (Norberg, 2019). This on-going study contributes to the fields understanding of the complexity of interplay between modes by using Engebretsen’s (2012) theory of the necessity of balance between cohesion and tension to analyse how modes are used when introducing new mathematical ideas in textbooks.

We have delimited our analysis to the introduction of equality and of subtraction in three commonly used textbooks for first grade students. Figure 1 shows an example of an introduction of equality. In this example, we identified one type of cohesion between the use of circles (images) in two groups, the numerals with an equal sign between (symbols), and the words “equal numbers” and “is equal to” in the yellow box. These modes address the same objects in the same way, that is, there is cohesion. At the same time, there are no symbols present in the exercise below. Students should only circle an equal number of objects found in the picture. Thus, there is a type of tension between the yellow box and the exercise since the different modes do not have the same role in these different parts. Further analysis will identify different types of cohesion and tension between modes, before examining how these are handled by students.

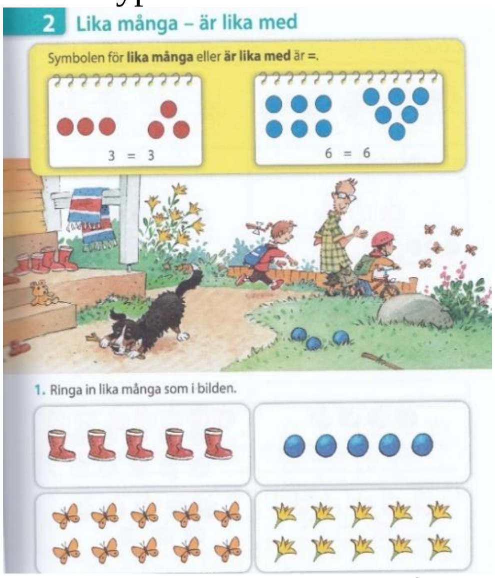

Figure 1. Introduction of equality (Mitt i Prick 1A, p. 9)

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COMPARING SELF-REPORTS AND KNOWLEDGE TESTS FOR ASSESSING MATHEMATICS TEACHERS' TPACK

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Teachers need technology-related knowledge to make effective use of digital technology in the classroom. Previous studies have often used self-reports to measure these knowledge facets. However, it is not very clear if self-reports are valid measures for this purpose.

This study investigates how mathematics teachers' self-reports correlate with their performance in a paper-pencil knowledge test regarding four facets of technology-related knowledge: content knowledge (CK), pedagogical-content knowledge (PCK), technological knowledge (TK), and technological pedagogical content knowledge (TPACK) (Mishra & Koehler, 2006). The items in both the questionnaire used for self-reports and the paper-pencil test focused on the specific domain of fractions. Participants were $N = 173$ pre- and in-service mathematics teachers. To assess self-reports, we adapted a questionnaire by Schmidt et al. (2009). We also compiled a knowledge test with four subscales, based on items from existing test instruments. The knowledge test was validated in a pilot study. All subscales had sufficient reliability (McDonald's ω_{CK} = .81, ω_{PCK} = .74, ω_{TK} = .85, ω_{TPACK} = .82, respectively).

The correlations between the scores in the self-reports and the paper-pencil test were low or very low for all subscales: CK (r = .23*, 95% CI [.08, .37]), PCK (r = .23*, 95% CI [.08, .37]), TK (r = .00, 95% CI [–.16, .15]), and TPACK (r = .13*, 95% CI [–.03, .28], *p < .05).

These results suggest that self-reported measures and paper-pencil knowledge tests may not always yield equivalent results. This raises concerns about the comparability of studies that use distinct measurement tools to assess knowledge. We recommend that studies should be more critical about whether they use knowledge tests or self-reports to assess teachers' TPACK.

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AN EVALUATION FRAMEWORK FOR EXPLANATORY VIDEOS IN FLIPPED MATHEMATICAL MODELLING EDUCATION

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As a well-established educational area mathematical modelling is strongly emphasized in many national curricula, and it is widely acknowledged that an important goal of mathematics education is to increase students' ability to deal with real-world problems. Contemporary mathematics curricula call for innovative methods and technological advances that can assist learners in promoting their modelling competence (Niss & Blum, 2020). Albeit, the application of digital technologies in mathematical modelling is still in its early stages, and emerging pedagogies including flipped classroom (FC) are rarely used to support modelling competency development. Empirical and theoretical studies on the usage of EVs in teaching mathematical modelling have been hardly carried out in the past despite the increasing relevance of new technology in mathematics education in general and in mathematical modelling education particularly. Considering this research gap, we carried out an empirical study on flipped classroom at tertiary level focusing modelling education at a large German university and developed a framework for quality criteria of EVs of modelling. Within different seminars at master level we focused on the design characteristics of FC pedagogy, developing and using EVs, and the key concepts of mathematical modelling. Within the empirical study we explored the perspectives of 42 of pre-service teachers on the advantages and pitfalls of FC pedagogy and EVs in modelling education through online surveys and interviews. As part of the seminar, pre-service teachers created their own EVs to teach modelling, which we analysed concerning the key characteristics of these videos based on our evaluation framework of EVs in teaching mathematical modelling (Cevikbas, under review). The results showed that the majority of the participants appreciated the usage of FC and EVs in modelling education and saw many advantages. Further results revealed that EVs and FC might offer (pre-service) teachers innovative ways to engage their students in learning modelling after some start-up difficulties concerning creating the content and technological requirements have been resolved. Overall, the present study provides interesting insights into innovative approaches for modelling education, which need to be explored in further studies.

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INTEGRATING CORE COMPETENCIES IN INSTRUCTION OF PRIMARY MATHEMATICS

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The purpose of this study is to identify supports that Kenyan primary mathematics teachers need to effectively integrate core competencies, the 21st century skills, with mathematics content. This is a new phenomenon yet fundamental to the current generation of school children in the 4th industrial revolution era. According to Remillard and Heck (2014) a curriculum is a process that involves transformation of the official curriculum thus the teacher's intended and enacted curriculum is the teacher's interpretation of it. As such, this study used the construct of teaching presence (Rodgers, 2006) as its framework. The guiding research question is—what support do mathematics teachers need to enable them to develop the knowledge and skills to implement the core competencies in mathematics classrooms with fidelity?

This is a descriptive study, and 16 participants were sampled. Data were collected from 88 classroom observations of about 40 minutes per lesson and 63 interviews of about 10 minutes each and analysed via coding and categorizing emerging patterns.

The findings of the study show that all the participants were aware of the core competencies as outlined in the Kenyan curriculum. Communication and collaboration competencies were implemented the most, followed by problem solving. The rest of the competencies e.g., self-efficacy, were minimally employed or were missing. There was no evidence of rigor for the deployed core competencies. Participants superficially perceived communication as mere talk, collaboration as working in small groups and problem solving as answering questions as modelled by the teacher.

The study concludes that primary mathematics teachers need support on the meaning, knowledge, and processes of each of the core competencies with exemplification. Additionally, teachers need guidance on how to choose and implement challenging non-routine tasks that deepen the enactment and development of core competencies such as problem solving, critical thinking, creativity, and imagination.

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HOW DOES A FACILITATOR SUPPORT TASK DESIGN DURING SCHOOL-BASED LESSON STUDY?

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While lesson study (LS) effectively promotes teacher learning (e.g., Fernandez & Yoshida, 2004), there are challenges, such as less interest in task design and difficulty providing continuous support outside Japan (Fujii, 2018). Given the challenges faced, one solution could be to pay attention to the activities of the facilitators, who are crucial to sustaining and encouraging teachers during school-based LS in the face of newcomers every year. However, there is limited research focusing on it (Lewis, 2016).

This study explores an experienced Japanese facilitator's intention and reflection who participates called kenkyu shunin in a school-based LS. With the background described above, the research question is, what facilitator insights support task design that participating teachers need during LS?

Data were collected by participant observation and recordings taken from April 14 to November 14, 2022, in a public elementary school-based LS, in Japan. Data sources included verbatim transcriptions of video-recorded meetings, audio-recorded facilitator interviews, and reading material distributed such as lesson plans. Through thematic analysis (Braun & Clarke, 2006), codes were extracted to clarify a facilitator's intention and reflection to support task design and improve mathematics lessons.

From this analysis, a facilitator has asked to review the results of the last lesson study and reconsider the research hypothesis and specific procedures by the actual situation of the learners, providing teachers with an overview of the new curriculum in the lesson plan phase. He intended for teachers to verify that and consider differences in mathematical content from previous lessons, especially in task design, realizing the different support needed for teachers having different teaching experiences such as using mathematical representations in the observed lesson, post-discussion phase.

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PSLE MATHEMATICS TASKS AS IMPETUS FOR TEACHERS' REFORM IN THEIR INSTRUCTIONAL PRACTICE

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Big Ideas, including proportionality, are central to mathematics as they connect ideas coherently from different strands and levels thereby facilitating a deeper and more robust understanding of individual topics in mathematics. Research has documented that teachers' lack of relevant content knowledge of Big Ideas in mathematics translates into their lack of explicit attention to Big Ideas underpinning mathematics taught in schools and results in developing isolated compartments of mathematical knowledge in their students (Askew, 2013). A research study, Big Ideas in School Mathematics (BISM) is presently underway in Singapore and a part of it is situated in two primary schools where teachers are undergoing professional development (PD) led by mathematicians and mathematics educators.

As part of the work in the schools, mathematics teachers were involved in solving some past Primary School Leaving Examination (PSLE) mathematics tasks that encapsulated the big idea of proportionality. Following which, in their year groups, they reflected on how their instructional practices facilitated or could facilitate the development of proportionality amongst learners. One such task was:

PSLE 2021 Mathematics paper 1 question 15 (Ministry of Education, 2022)
A box contained brown and green balls. 40% of the balls were green. After some yellow balls were added to the box, 26% of the balls were green. What percentage of the balls in the box were yellow? (1)14%   (2) 34%   (3) 35%   (4) 39%

A group of teachers in one of the schools drew on their experiences of solving the above task, reflected about their instructional practices and planned a lesson for their grade six students on the topic of percentages. The lesson was enacted and video recorded. The video-record of the lesson was reviewed by the teachers and professors during a meeting. A key finding was that teachers hands-on work involving the assessment task led them to reflect on their past instructional practices and make deliberate plans for reform in their practice, i.e., illuminate big ideas where possible during instruction.

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AUSTRALIAN MATHEMATICS TEACHERS' KNOWLEDGE, BELIEFS AND ATTITUDES TOWARDS MANIPULATIVES

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Manipulatives (or concrete materials) are frequently used to teach mathematics in lower primary years, but their use declines as students age, and they are nearly absent in secondary school (Carbonneau et al., 2013). In addition, while research demonstrates their effectiveness in primary school, few studies have examined their effectiveness in mainstream secondary classes (Carbonneau et al., 2013).

Given that the revised Australian Curriculum (Australian Curriculum Assessment and Reporting Authority [ACARA], 2022) endorses and encourages using manipulatives in secondary mathematics education, this study examined Australian secondary mathematics teachers' attitudes and beliefs towards these tools. The primary research question was: what beliefs do secondary mathematics teachers hold about the benefits and challenges of using manipulatives in their teaching?

A survey design was employed to explore teachers' attitudes and beliefs towards manipulatives and collected data from 51 responses recruited through Australian Facebook groups targeted at secondary mathematics teachers.

Whilst acknowledging the benefits of manipulatives in improving students' conceptual understanding and encouraging collaboration and engagement, most respondents did not use them. The reasons cited were that they lacked training in using and integrating these materials, particularly for higher-level topics like algebra. Additionally, managing these materials in the classroom posed challenges. Surprisingly, the respondents preferred concrete rather than virtual manipulatives, even though the research was conducted after an extended period of online teaching due to COVID restrictions.

The study contributes fresh perspectives to an expanding literature on using concrete materials in mainstream secondary mathematics education. The presentation will explore the findings in detail and present ongoing research.

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ENGAGING ALL LEARNERS BY RECRUITING AND RETAINING MINORITZED MATHEMATICS TEACHERS

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The primary purpose of this research was to provide insight into the narratives and experiences of minoritized preservice teachers (PSTs) interested in teaching mathematics and how those experiences impact recruitment and retention of those minoritized teachers. Narrative inquiry framed through equity and identity lenses were chosen as the methodology for gathering and analysing data from one-on-one, structured interviews as well as a focus group interview. A finding from the data analysis resulted in one specific resonant thread, impact as a minoritized teacher.

Researchers have identified how minoritized mathematics teachers facilitate the learning and doing of mathematics for not only minoritized students but all students. Cherng and Halpin (2016) found that minoritized teachers may have the ability to navigate racial stereotypes about academic achievement and can equip students to combat these stereotypes. These teachers also act as positive role models who have broken racial stereotypes and improves student perception of mathematics as doable. This in turn, reduces the mathematical anxiety for some students.

The data analysis of the interviews resulted in a resonant thread, impact as a minoritized teacher. As all participants resided in the south-eastern United States, they took those experiences to heart because they never saw themselves as future mathematicians as they did not see examples of successful minoritized mathematicians through their instructors' race and ethnicity. The mere representation of minoritized mathematics teachers can have a lasting impact on all students, but especially, minoritized students.

One participant, Gabriel, mentioned going to a school for a practicum and having students walk up to him and say, "hey, you are Brown like me". This comment fuelled Gabriel to continue to pursue mathematics education because he knew the impact he could make by the mere representation of a minoritized teacher. Gabriel also was worried about his Latinx representation to those that were not minoritized, especially the parents. He feared he would be challenged about his content knowledge due to his minoritized status. Gabriel knew he needed to teach in order for non-minoritized communities to see his positive impact in mathematics classrooms as a minoritized mathematics teacher.

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A TASK DESIGN FRAMEWORK TO TRANSFORM GENDERED VIEWS OF MATHEMATICS

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The perceptions such as ‘mathematics is for boys’ have permeated people’s awareness and identified as a problem. In Japan, although there is no gender difference of mathematics performance, it has stated ‘calculation skills’ and ‘logical thinking skills’ tended to be considered masculine skills (Ikkatai, et al., 2021), which are referred to as gendered views of mathematics here. Acquiring the experience of ‘doing mathematics’ through the tasks is important for students to transform these views of mathematics. This paper aims to develop a framework of task design to transform these views of mathematics in classroom. The task is defined as a teaching material aligning with Japanese national curriculum.

As a foundational study, the framework is drawn from the interpretation and conceptualisation of literature. This paper proposes a framework of task design based on ‘women-centred mathematics’ (Kaiser & Rogers, 1995) to incorporate ‘feminine attributes,’ and diverse and different voices, especially at high school level.

In conclusion, the task design framework includes 1) addressing diverse mathematics associated with the context of family/work as a female domain and the achievements of male/female mathematicians, and 2) valuing the process of mathematics as a ‘connected’ procedure. It intends not to enhance the female domain, but a challenge to our divided perceptions of the male/female domain. It might allow individuals to make choices and take action that are not constrained by masculinity/femininity.

ACKNOWLEDGMENT

This study is supported by Japan Science and Technology Agency (JPMJSP2124).

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DESIGNING OPPORTUNITIES FOR MATHEMATICAL LEARNING AND COMPUTATIONAL THINKING THROUGH FAMILY STORIES

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This short oral focuses on a project that sought to build on family stories to support children's integrated learning of mathematics and computational thinking. The research conducted analyzed observational notes of the family workshop sessions, student presentations, and families' dioramas to understand opportunities to learn about math and computational learning. We focus on the design of the sessions in creating a hybrid space that supported community and voice during the COVID-19 pandemic.

In this project, we sought to answer the question of how to make mathematics and computational thinking meaningful to children and their families. We worked with families as part of a six-session coding workshop, conducted face-to-face and then transitioning to a virtual format. Sessions 1 and 2 introduced coding through the use of hands-on activities and pictures that helped unpack common terms (e.g. algorithm, loop, etc.). Sessions 3 and 4 introduced Scratch, and sessions 5 and 6 focused on families coding and adding robotics to their dioramas which featured a scene from their stories. Our approach was inspired by the *Techtales* curriculum. In this analysis, we draw on sociocultural perspectives that view learning as shifts in participation reflected in individuals' experiences and changes in trajectory (Nasir, 2012).

For this analysis, we drew on observational notes from each workshop session, researcher's reflective notes, and the discourse around the dioramas that were created. We analyzed the notes for critical incidents related to design decisions regarding the learning sessions. Our findings indicate three themes: First, the learning sessions seemed to shift from story to coding rather than a true integration during sessions. Second, it seemed that more structure was needed to support learning about coding. Third, the children involved were able to unpack and explain the code that was used in relation to the story. This research has implications for how we design learning opportunities that leverage family practices and identities.

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FUNDAMENTAL ABILITIES OF JAPANESE STUDENTS FOR LEARNING MATHEMATICAL PROOFS IN GRADES 5 TO 7

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Japanese students begin to learn mathematical proofs in earnest from the grade 8 (13-14 years old) in junior high school. However, it has long been pointed out that students' achievement is not sufficient (e.g., Kunimune, 2017). To improve this situation, it is necessary to consider a smooth connection of learning activities from elementary school to grade 7 in junior high school to create a suitable foundation for studying of mathematical proofs in grade 8. However, the actual situation of students' abilities, which is essential information considering the characteristics of students' descriptions when they freely write explanations, have not been sufficiently clarified. This study focuses on learners from grades 5 to 7, in particular, with an aim to clarify the characteristics of their descriptions as fundamental abilities for learning mathematical proofs. Two questions were developed ("area problem" and "angle problem") in which students were asked to freely describe their judgments of the validity related to basic figures. A questionnaire survey of a total of 752 students (G5 = 168, G6 = 160, G7 = 138, G8 = 155, G9 = 131) in Japanese public schools was conducted. Furthermore, an analytical framework based on the theoretical two viewpoints, "Using of Characterizing Properties" (Steiner,1978) and "Functional Explaining" (Wilkenfeld,2014) was considered, and data were analyzed both quantitatively and qualitatively using this framework. Findings revealed that there was no improvement from grade 5 to 6 in "describing the basis of their judgment by relating it to the properties of figures", and this emerged as a common trend for the two problems. The percentage of learners who could do this increased from grade 6 to 7. However, at the end of grade 7, about 30% to 40% of the students were unable to do so. This result suggests that these students may not have developed the fundamental ability to begin learning mathematical proofs by the start point of the learning. In the presentation, I will present the details of the students' description of the two problems and discuss the study findings.

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INVESTIGATING THE IMPACT OF NEWSPAPER PREPARATION METHOD ON THE TECHNOLOGICAL PEDAGOGICAL CONTENT KNOWLEDGE OF PRE-SERVICE MATHEMATICS TEACHERS

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The teacher-centered approach usually makes the philosophy and history of mathematics courses less appealing, and thus, pre-service teachers (PTs) graduate without learning how to use the knowledge they have learned in their professional lives. (Türker et al., 2015). Considered within the scope of 21st-century skills, this situation falls within technological pedagogical content knowledge (TPCK) (Mishra & Koehler, 2006). In this study, a new approach, named the newspaper preparation method (NPM), was developed for teaching the philosophy of mathematics and aimed to examine the effect of this method on the TPCK of PTs. To investigate the impact of the NPM on PTs' TPCK, the exploratory design, one of the mixed methodological designs, was used. In the quantitative part, data were collected using the "TPCK Scale with Pre-Service Mathematics Teachers" developed by Önal (2016) was used and analysed dependent sample t-test. Qualitative data were collected with a semi-structured interview form -prepared by the researcher- and analysed using by content analysis technique. The study sample consists of 40 PTs studying at the department of primary mathematics education. From the pre- to post-test there was a significant increase as for that the TPCK. Also, PTs realized they acquired teamwork, socializing, and authorship skills. Based on the results NPM has converted the class from a dull program into a motivating and entertaining part of the curriculum. These results indicate that the NPM positively affected the PTs' TPCK. It is predicted that this study will be a source of inspiration for those who teach the philosophy and/or history of mathematics.

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THE ROLE OF CONSIDERING STATISTICAL VARIATION IN DATA-BASED ARGUMENTATION: AN EXPLORATORY STUDY

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Statistical variation makes drawing conclusions from data complex. As enabling participation in discourse based on data has been identified as an important goal of statistics education (e.g. Gal, 2002) from early schooling on, students should be enabled to evaluate claims by specifically referring to available data, even if the data set is marked by statistical variation (e.g. Krummenauer et al., 2022; see also Ben-Zvi & Makar, 2016) – we refer to such requirements by the notion of *data-based argumentation* (e.g. Krummenauer et al., 2022). Despite the developing literature on early statistics education, research on young primary students' data-based argumentation and their difficulties is scarce. Consequently, this study addresses this research need with a focus on the following questions: *Is it possible for beginning first-graders to develop data-based arguments if the data is marked by statistical variation, and can cases of answers provide insight into the nature of students' difficulties?* Both qualitative and quantitative analyses were carried out in order to explore interview data from N = 29 beginning first-graders in this regard. The results imply that considering statistical variation can provide a high level of complexity to young students, but also yield insights into students' particular difficulties and how students could be supported by their teachers to overcome these difficulties. The study, therefore, contributes to a deeper understanding of young primary students' competence of data-based argumentation and implies possibilities of fostering students' learning in this regard from the beginning of the primary mathematics classroom on.

Acknowledgements. The phase of data collection of the reported research had been supported by research funds from the Senate of Ludwigsburg University of Education.

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IS TEACHER NOTICING OF STUDENTS' THINKING SUFFICIENT?

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Teacher noticing of students' thinking has received increased awareness in the mathematics education discourse, as it is often considered as key for adaptive teacher reactions (e.g., Sherin et al., 2011). Nevertheless, classroom situations are complex, and research suggests that there is a need to take a more comprehensive look at teachers' noticing of student thinking. One approach can be described with *multi-criterion noticing*, referring to a spectrum of criteria that can support pre-service teachers' (PTs) analysing when making sense of classroom interaction (Kuntze et al., 2021). Building on multi-criterion noticing, we examined PTs' noticing of rather non-adaptive teacher reactions together with noticing of students' thinking. Our research question is: *Is successful noticing of students' thinking sufficient for PTs to notice also whether teacher reactions to learners are adaptive to their needs?*

This study focuses on analyses made by 26 PTs for a classroom situation presented as cartoon vignette. The PTs' analyses were coded using qualitative content analysis. Results yield in-depth evidence of frequent answers in which the non-helpfulness of the vignette teacher's reaction was not noticed by the participants despite their noticing of key aspects of students' thinking. The findings suggest that noticing of students' thinking does not necessarily lead to noticing of the teacher's non-adaptive reactions related to learners' thinking and needs. We conclude that multiple foci of noticing, e.g., on *relating different mathematical approaches of the participants in the interaction*, merit increased emphasis in mathematics teacher education and further research.

Acknowledgements. This study was supported by the project coReflect@maths (www.coreflect.eu), which was co-funded by the Erasmus+ Programme of the EU. The European Commission's support for the production of this publication does not constitute an endorsement of the contents, which reflect the views only of the authors, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

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PSYCHOMETRIC EVIDENCE OF THE BRAZILIAN MATHEMATICS ANXIETY SCALE FOR PEDAGOGY STUDENTS

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Understanding difficulties in teaching and learning Mathematics, and identifying possibilities for improving teacher training, require characterizing experiences of anxiety concerning mathematics in the academic history of future teachers. Our study aimed to investigate psychometric evidence of the Mathematics Anxiety Scale version for Pedagogy university students (EAM-Ped), professionals who will work in the early years of Elementary School.

This study was carried out in two stages. In Stage 1, we investigated the content validity (CV) by using three experienced researchers from the field of Mathematics Education, Psychology and Education that participated as judges and, for the investigation of the semantic validity (SV), we had four students from the target population as EAM-Ped evaluators. In Step 2, we investigated the validity of the internal structure, and 395 Pedagogy students were enrolled. The EAM-Ped has 35 items composed of situations that assess, based on a Likert scale, the feeling and behavior experienced in subjects for teaching mathematics.

The results indicated evidence of content validity (CVCs for clarity, relevance and practical relevance above 0.86) and semantics validity (CVC Clarity = 0.86). We identified evidence in favor of conducting Exploratory Factor Analysis (EFA, see Damásio, 2012): significant p-value in Bartlett's sphericity test and a result of 0.96 in the KMO test. To conduct the EFA, we performed a Parallel Analysis based on a Pearson correlation matrix, which indicated the retention of one factor. This was confirmed by UniCo (0.99), ECV (0.93) and MIREAL (0.20). In the EFA, we found an explained variance of 70% and factor loadings greater than 0.51, with 31 items having loadings greater than or equal to 0.60. We also observed preliminary evidence of reliability (McDonald's Omega = 0.97). The results were promising and support further studies with the EAM-Ped.

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THE USE OF 3D DESIGN AND PRINTING ACTIVITIES TO DEVELOP MATHEMATICAL MODELLING COMPETENCIES

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The use of digital technologies has been identified as an effective strategy for fostering students' mathematical modelling competencies. However, while mathematical modelling learning activities have been extensively evaluated in paper-and-pencil settings, they were scarcely researched in technology-enhanced settings (Cevikbas et al., 2022). In this study, we examined the use of 3D design and printing (3DP) activities to foster middle-school students' mathematical modelling competencies. Following the learning-by-design approach (Kolodner et al., 2003), our students performed design tasks through iterations in two activity cycles, "Design/Redesign" and "Investigate & Explore", connected by the needs to do and know.

The purpose of our study was to identify the characteristics and effectiveness of 3DP activities for developing mathematical modelling competencies in middle school students. The sample included 112 students aged 11 to 14, of whom 88 participated in 3DP workshops. A mixed method was adopted to gather and analyse data from observations, audio-recorded discussions, students' artefacts, questionnaires and pre and post-tests.

Results revealed that during the 3DP activities, the students practised several mathematical modelling competencies: they identified the aspects of the design task that could be solved mathematically, translated the design problems into mathematical ones, devised mathematical solutions, incorporated them into design solutions, and validated them by inspecting the resulting designs and 3D-printed artefacts. Our study suggests that authentic 3D design tasks may facilitate the development of mathematical modelling competencies through the interaction of design and modelling cycles and provide context for real-world mathematics applications. In this presentation, further results will be discussed.

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THE RELATIONSHIP BETWEEN SUBJECTIVE NORM AND MATHEMATICS LEARNING INTENTION: A PLANNED BEHAVIOR MODEL FOR ELEMENTARY SCHOOL STUDENTS

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Human learning is a process of continuous interaction between individuals and their social environment. Based on social psychology, the theory of planned behavior (TPB) emphasizes that an individual's intention to engage in a specific behavior is an immediate determinant of whether the behavior occurs (Ajzen, 1991). The intention will be affected by an individual's attitude towards the behavior, subjective norm, and perceived behavioral control. Significant others such as parents and friends influence a student's mathematics learning (e.g., Brown & Larson, 2009; Eccles, 2007). Therefore, exploring parents' and friends' understanding of "what matters" in mathematical thinking and learning will be helpful in students' mathematics learning. Based on TPB, this study investigates the relationship between the subjective norm formed by parents and friends with the mathematics learning intention (MI). This study edited the scales of mathematics interest, subjective norm, perceived control, and MI, then recruited 391 fifth and sixth graders to participate. We used Mplus software to conduct confirmatory factor analysis and structural equation modeling. The analysis results showed that all the scales developed in the study had good reliability and validity. The regression (structural) coefficients for the subjective norm second-order factor model were high for parents' subjective norm (γ =.702; $p < 0.001$) and low for friends' subjective norm (γ=.295; $p < 0.001$). It suggested that the main source of the subjective norms was the parents of elementary school students. There were significant positive relationships between mathematics interest and subjective norm with MI, while the relationship between perceived control with MI was positive but not statistically significant. This study suggested relevant issues in the future. For example, future research could focus on older adolescents to investigate changes in the association among the constructs of TPB, especially for the subjective norm and MI during adolescence.

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EXAMINATION OF DIFFERENT COGNITIVE-STYLE STUDENTS ON SOLVING MATHEMATICS PROBLEMS

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Many researchers have examined cognitive styles of individuals as field-dependent (FD) or field-independent (FI) and the relationship between cognitive style and performance in different subject domains (e.g., mathematics) (Morris et al., 2019). In this study, we aimed to investigate FD and FI students in solving a variety of mathematics tests using the event-related potentials (ERP) methodology. A total of 155 Taiwanese high school students were required to take the Hidden Figure Test (HFT) in order to determine if they were FD or FI cognitive styles. Those students were also required to solve the simple multiplication test, the geometry area test, the function test and the insight-based test. The behavioral analyses showed that FI students performed significantly better than FD students in the four mathematics tests. FI students had higher response accuracy (Acc) than FD students across the four tests. FI students also answered the problems significantly faster than FD students in the multiplication and the geometry tests. With respect to ERP data, the analysis of the P1 component showed that FD students had significantly higher amplitudes than FI students on the geometry test but not the remaining three tests. The finding indicated that FD students paid greater attention to and involved higher cognitive efforts in perceiving information stimuli than FI students when solving geometry problems. With respect to the P3 component, only the geometry test caused significant differences between FI and FD students but not the other three tests. FI students had higher cognitive loads in the central area of brain than FD students. On the contrary, FD students had higher amplitude in the occipital area of brain than FI students. Regarding the remaining three tests, no significant difference in the brain areas was found. With respect to the analysis focusing on the hemispheric lateralization, we found that FI and FD students performed a similar pattern of hemispheric differentiation in all the mathematics tests, which conflicts with the hypothetical theory of hemispheric lateralization as shown in the literature. Our study indicated that the brains of FI and FD students might function in similar ways when solving different mathematics problems. Details of the analyses on the behavioral and ERP data and research implications will be discussed during the conference presentation.

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PRESERVICE TEACHERS' EPISTEMOLOGICAL BELIEF ON CLOSE-ENDED AND OPEN-ENDED PROBLEM SOLVING

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Ernest (1989) categorised teachers' beliefs about mathematics into three different types: Instrumentalist, Platonist, and Problem-solving view, and Beswick (2005) and Safrudiannur & Rott (2021) further developed the structure for teaching\learning mathematics and mathematical problem solving. Although it is widely discussed that the open-ended tasks have potential benefits for mathematics classrooms, we know less about preservice teachers' perspectives on the open-ended tasks and their connection to their epistemological beliefs.

The participants in the current study include 84 preservice teachers (aged M=23.54 SD=1.417, 9 males) in the same master program from three normal universities located in different areas of mainland China. The instrument is revised from the **t**eachers' **b**eliefs on **t**heir **p**ractice (TBTP, by Safrudiannur & Rott, 2021) by adding descriptions and items to the close/open-ended problems. TBTP includes rank-then-rate items and consideration of students' abilities.

Overall, our findings on teachers' beliefs are similar with Safrudiannur & Rott (2021). Through Wilcoxon signed-rank tests, the significant differences between teachers' beliefs about teaching close-ended and open-ended problems are found in the Platonist view within both the high-ability classes (Z=2.471, p=0.013) and the low-ability classes (Z=3.771, p<0.001), indicating preservice teachers seem to hold relatively more content-understanding perspective (Platonist view) in the close-ended problems than in the open-ended problems. After analysing the correlation through Spearman's rho tests, more significance and larger effect sizes take place in the open-ended problems than in the close-ended problems, indicating the open-ended tasks probably have more potential for teachers to implement their own beliefs about the nature of mathematics.

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PRACTICES ON STATISTICAL INFERENCE PROPOSED IN THE CHILEAN MATHEMATICS CURRICULUM

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Statistical inference has been incorporated into the mathematics curriculum of secondary education in Chile. In this case, formal topics of statistical inference are studied in the third and fourth grades of secondary education (17 and 18 years), and students are expected to have the ability to make inferences using hypothesis tests and confidence intervals (Mineduc, 2019). However, previous studies have shown that the teaching of statistical inference in educational institutions focuses on the application of formulas and rules to make an inference, as well as numerous errors of understanding and application of inference techniques (Lopez-Martín et al., 2019).

In this oral communication shows how inference is introduced in the secondary education curriculum of Chile and the correspondence with the Levels of Inferential Reasoning. For this, we employed the theoretical-methodological tools of the OntoSemiotic Approach and for the analysis of the treatment given to statistical inference the mathematics school texts (six student text and six activity book) of secondary education (12 to 18 years old) were selected. Thus, based on the activities and practices suggested by the Chilean mathematics curriculum, we have determined that the characteristics of the inferential reasoning that the curriculum is intending to promote are consistent with some of the indicators of the levels of inferential reasoning, mainly those that make up the *Level 1* (informal inferential reasoning) and the *Level 4* (formal inferential reasoning).

Finally, we would like to highlight our concern regarding the treatment given to statistical inference in textbooks and activity workbooks, given the role of the textbook in mathematics classes, with respect to two aspects: (1) the jump from informal to formal inference and (2) the fact that the activities proposed in textbooks focus more on statistical formulas and rules than on inferential reasoning. In this sense, the levels of inferential reasoning could be used to progressively operationalize the study of statistical inference in the Chilean secondary education curriculum.

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TEACHER EXPLANATIONS OF GRADE 8 OPERATIONS ON INTEGERS

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Many teachers struggle in handling the topic of operations on directed numbers. Often, they teach memorisation of the facts without adequate explanations to learners. Research has highlighted that teachers are often challenged when asked 'to reason conceptually about integers or apply integers to various contexts' (Wessman-Enzinger & Tobias, 2020; p.2). Based on that, I was motivated to examine if teacher professional noticing on learner mathematical thinking during the teaching of grade 8 directed numbers and the use of models for teaching directed numbers sourced from literature can transform their practices. Firstly, in a qualitative design study, lessons on the teaching of the topic were observed and learners' productions in class work were diagnosed to interpret learners thinking. Then the teachers underwent professional development led by the researcher on how to attend, interpret and respond to learner offerings in class contingent and expedient to situations. At the same time, the teachers were exposed to models for teaching direct number operations sourced from literature. Teacher interpretations of learner thinking informed teaching intervention lessons which the researcher observed. Also, in the teaching interventions the teachers were encouraged to utilise models for teaching operations on integers earlier shared. In the third stage I determined the possible effect of the teaching intervention on their practice.

The study showed observable shifts in teachers' professional knowledge for teaching directed numbers. The models they used helped them to think about operations on directed numbers in several ways.

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ENHANCING TEACHERS' PCK FOR MULTIPLICATIVE THINKING TO SUPPORT MATHEMATICS LEARNING

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Multiplicative thinking forms the basis for understanding proportions, patterns, fractions, measurement, rates, percentages, statistical thinking, the development of algebraic thinking and understanding the complex issues in society. However, student performance in this area appears to be low (Siemon, 2013). Teachers' Pedagogical Content Knowledge (PCK) is critical in determining students' learning attainment in this area.

This study employs an embedded mixed methods approach to investigate in Australia 62 primary school teachers' PCK for developing multiplicative thinking in students. These teachers range from Pre-service teachers to Novice, Experienced, and Expert teachers. A questionnaire was used to yield quantitative and qualitative data. The three key teaching stages for developing multiplicative thinking provide the focus for this study, with these stages being: (1) Transition stage (from additive to multiplicative thinking), (2) Multiplicative stage (multiplication and division word problems), and (3) Proportional reasoning. Zhang & Stephens (2013) model of Teacher Capacity provided the framework for instrument design and data analysis.

The results show limited teacher knowledge about the connection between multiplicative thinking and other key topics in mathematics. This study also found that teaching experience influenced teachers' PCK for multiplicative thinking across the three key teaching stages. For instance, Experienced and Expert teachers demonstrated a higher PCK for multiplicative thinking at the Transitional teaching stage compared to Pre-service and Novice teachers. Higher PCK was evident among the Pre-service and Novice teachers at the multiplicative and proportional teaching stages compared to Experienced and Expert teachers. The findings complement current efforts to enhance teachers' capacity to support students' learning around multiplicative thinking and highlight specific areas requiring attention in teacher professional development and teacher preparation programs.

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PATHWAYS OF ATTENTION: TECHNOLOGY AND TENSIONS

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We examine shifts in attention (Mason, 1998) of a prospective teacher, Lilac (pseudonym), that emerged from her engagement with a computational modeling task focused on issues related to climate change. Research conducted with teachers who incorporate social justice issues into their mathematics teaching has highlighted tensions related to navigating between the social issue and the mathematical content (e.g., Bartell, 2013). Shifts in attention can be provoked by experiences of tension or disruption (Mason, 1998), and in this presentation we consider the research question: How can we understand the role of technology in influencing a prospective teacher's attention during her exploration of interdisciplinary connections related to modelling climate change issues with mathematics?

We analyse Lilac's engagement with a task focused on collecting and modelling climate-related data using the micro-controller, the BBC micro:bit©. The micro:bit is an electronic device that attaches to a computer and includes built-in and external sensors that allow users to collect and analyse real-world data such as temperature and soil moisture. Our objective for exploring climate issues through coding with micro:bit was to leverage the technology's capacity to collect and model real-world data. We sought to provide accessible opportunities through coding to develop rich understandings and experience new pedagogies (e.g., Gadanidis, 2015), while also demonstrating best-practices for incorporating coding into mathematics learning (Gleasman & Kim, 2020). We found that Lilac's pathway of attention was influenced by the physicality of the technology and tensions around the use of electronics, and these tensions diverted her from the original purposes of the task. We offer recommendations for instruction and research in teacher education.

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CHANGES IN TEACHERS' CONCEPTIONS WHILE USING MULTIPLE OUTCOME TASKS

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This study is performed in the framework of the Math-Key program (Leikin et al, 2023), the goal of which is to develop students' mathematical creativity along with their mathematical knowledge and skills. The program introduces mathematical tasks that have multiple solution outcomes (MO). The tasks are accompanied by electronic applets that reflect each task's structure and allow students to explore (Pitta-Pantazi, 2023) the situation given in the task. The tasks are unconventional and thus require from teachers meaningful didactical change.

The study presented here analyses changes in teachers' conceptions about the role of a multiplicity of solution outcomes and the use of explorative dynamic applets in mathematics education. Nine middle-school teachers who implemented Math-Key tasks in their classes were interviewed before and after the implementation. The teachers changed their conceptions about the goals that can be attained using Math-Key tasks, from giving students an opportunity to think to developing various types of thinking and changing teaching approaches. The unconventionality of the tasks, which initially was viewed as a major pitfall, became, in the teachers' opinion, a major advantage, as the tasks' unconventional nature provoked students' curiosity and increased their enjoyment. Teachers initially considered the tasks to be appropriate for excelling students only, whereas after the implementation they argued that the multiple solutions and applets served as effective didactical tools for addressing the heterogeneity of classrooms and inspiring the development of self-regulation in learning. These findings lead to new questions about major elements of mathematical instruction that influence teachers' conceptions. More evidence from students' learning is needed to explain the changes outlined above.

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MATHEMATICS CLASSROOM EXPERIENCES OF PROSPECTIVE PRIMARY SCHOOL AND SPECIAL EDUCATION TEACHERS

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Many students develop negative beliefs and negative affect towards maths. Mathematics education (and, particularly, teacher preparation to teach maths) faces a vicious cycle in which prospective teachers will be expected to promote their students' relation with maths while their own school experience may have been riddled with maths anxiety and negative beliefs and affect (e.g., Bekdemir, 2010). Breaking this cycle is particularly important for prospective primary school and special education teachers, who work with children at critical ages in their learning process.

We conducted three focus groups with students of these two programs (total N = 12 students) at a Chilean public university. These students were in their third semester and, in the previous year, they attended two maths courses that mixed students from both programs. The conversation investigated their experiences with maths in school and in their teacher preparation programs. While students of the primary program reported more positive school experiences with maths than those of the special education program overall, students from both programs reported living impactful negative experiences in the school maths classroom, such as being shamed in front of the class because of not knowing the multiplication tables or irate teacher reactions to students' requests to re-explain some things. In contrast to the negative school experiences, several students felt that the university courses (more focused on processes than results) helped them to repair their relationship with maths. Students were aware that, as future teachers, they would be responsible for presenting maths to the next generations in a friendlier manner. Altogether, these results highlight the need for teacher preparation programs to actively address their students' relationship with maths as well as for more research on the reasons behind students' and prospective teachers' negative affect towards maths.

Acknowledgments. This research was part of a larger project, supported by the funds UOH/FID-fidc03, ANID/Milenio NCS2021_014, and ANID/Basal FB0003.

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VARIATION OF TEACHERS' EXAMPLE USE WITHIN A MATHEMATICS DEPARTMENT: AFFORDANCES FOR LEARNING IN DIFFERENT CLASSES IN THE INTRODUCTION OF FUNCTIONS

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Goldenberg and Mason (2008) argue that the examples which a teacher chooses and uses directly influence a student's emergent example space on a topic. Similarly, Marton (2015) suggests that the sequencing and enactment of examples can impede or facilitate students' discernment of critical aspects of mathematical knowledge, and how students can use this knowledge in the future. Content taught in classrooms does not necessarily translate to what is learnt but is nonetheless critical to the learning process (Adler, 2017). Thus, a teacher's use of examples contributes toward the quality and depth of their students' mathematical knowledge.

At an urban, private South African secondary school, different mathematics teachers instruct classes in the same grade. Although the same topic is covered simultaneously across classes, each teacher has the agency to independently select examples for their class lessons. This study examines the examples selected and used by different teachers in this department and explores the affordances for learning offered to students in the different classes. To collect data, three teachers in this department were observed and video recorded teaching their two introductory lessons on functions in Grade 10. Each teacher was also interviewed regarding their example use to provide information relating to the intended object of learning. The teachers' selected examples and the perceptible mediation of the examples were analysed according to Marton's (2015) variation theory. The examples were first analysed in and of themselves, and secondly, considering the associated teacher actions in their mediation in the classroom.

Preliminary findings suggest that the affordances for learning created in these three classes differ regarding the mathematical content of examples and the critical aspects of functions that were emphasised. This has implications for these students' future learning and for mathematics departments that function according to similar *modus operandi*.

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DO STUDENTS VALUE MODELLING WITH EXPERIMENTS?

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Modelling is a central competence in mathematics. Previous research has shown that students value modelling problems less than other mathematical problems (Krawitz & Schukajlow, 2018). One approach to foster students' value of modelling is to combine modelling tasks with experiments. With experiments students collect own data that are used for the subsequent modelling task. Thus, we developed a task involving an experiment measuring decay of beer froth, followed by modelling the collected data. Students' values regarding modelling were conceptualized following the expectancy-value theory (Dietrich, Moeller, Gou, Viljaranta, & Kracke, 2019). This theory focuses the importance of being good at a task or topic (attainment value), the intrinsic value as positive emotions concerning a task or topic, the utility value for future life and the costs (what resources of time or energy are needed to complete the task). In a pilot study, we investigate which values students report when solving modelling tasks involving experiments.

23 tenth-grade students of a grammar school worked on a modelling task with experiment. Afterwards they filled in a questionnaire (23 items, e.g., "The task exhausted me", Dietrich et al., 2019). The items were presented in a 6-point Likert scale (1=not at all true, 6=completely true). The reliability for every scale was acceptable (Cronbachs $\alpha > .63$) except for the utility value ($\alpha = .29$). The items of that scale address widely different fields of life. They cover utility for school career, for vocational career, for future everyday life. This wide range may be the reason for the insufficient low reliability. Furthermore, the sample is quite small for calculating the reliability. The mean of the attainment value is 3.26, of intrinsic value 4.39 and of costs 2.28. Against our expectation, we find astonishingly low mean values of all items for measuring utility value of 1.52 to 3.87. We can conclude that students seem not to see a significant use of modelling with experiments in their future, yet it provides a joyful experience. In the next steps, we will analyze whether students report different values when working on modelling tasks with and without experiments.

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DEVELOPMENT OF AN INSTRUMENT TO ASSESS STATISTICAL REASONING IN UNDERGRADUATE MECHANICAL ENGINEERING

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The research reported here focuses on assessment of statistical reasoning (e.g., Sabbag et al., 2018). A fundamental dilemma of assessment for postsecondary statistics is to be able to create meaningfully diagnostic tools that can assess statistical reasoning while dealing with constraints of large class sizes, disciplinary context, and varying mathematics abilities. (ASA, 2014).

An instrument containing eighteen items was developed utilizing items from a set of previously validated instruments. Contexts were changed to include typical mechanical engineering scenarios. In addition, items addressing multivariate relationships were written to specifically address conceptions related to the General Linear Model. Items were 5-response multiple choice items. The instrument was piloted in 5 sections of introductory statistics in a pre-post design. Two-hundred eighty-nine undergraduates completed both the pre- and post-tests.

The mean number correct for students in the pre-test was 5.85. Post-test performance showed a mean of 10.04 (Cohen's d = 3.27). This represents a mean gain of 4.2 items (s =4.3). Cronbach's alpha for the tool was near chance for the pre-test ($\alpha = 0.06$) and was high for the post-test ($\alpha = 0.80$), indicating that students' conceptions became more coherent over the course of instruction.

A 1*pl* Rasch model was applied to the data to assess item difficulty and its association with student ability. ICC plots centred close to 0 and ranged from about -5 to +5 standard deviations on student ability indicating good discrimination and high ceiling for the instrument.

Results show promise for the pilot version of this assessment. Continuing research is focusing on refining items and using the assessment to gauge the effectiveness of innovative instructional practices on students' learning of introductory statistics.

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USING PATTERNING TASKS FOR ASSESSMENT OF EARLY VISUAL LITERACY

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INTRODUCTION

Visual literacy is a complex concept encompassing the skill to convey ideas visually, form mental models and visual representations of concepts and their relationships, and use those constructs in reasoning (Trumbo, 1999). The ability to copy, finish, or describe patterns is considered to be part of early mathematical abilities related to the development of arithmetic and algebraic reasoning (Tsamir et al., 2017; Sarama and Clements, 2009; Mulligan & Mitchelmore, 2009; Papic, et al., 2011;) but it is also related to the development of early visual literacy. Preschool children most often encounter repeating patterns and spatial-structural patterns. Repeating patterns have a cyclical structure that can be generated by repeating the unit whereas spatial-structural patterns consist of shapes with invariant relations between different properties of geometric shapes (Papic et al., 2011).

Participants in this study, 192 kindergarten children in Serbia, were individually interviewed on recognition of the structure of the visual patterns' configurations and a unit of the visual repeating patterns. Findings suggest differences between children's responses based on the structure of the unit characterized by length, shapes, and color. In general, participants were more successful in solving tasks that involved patterns of shapes (in contrast to patterns of shapes and colors). The findings indicate shape as a primary visual identification of patterns. In this presentation, the results of the study

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SMARTPHONES: BRIDGING MATHEMATICS AND SCIENCE WITH NOVEL TECHNOLOGY

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Physics, an empirical science, relies on systematic data collection and its rigorous mathematical analysis. Yet the role of mathematics in physics is often taken for granted, thus concealing the complementarity of these two disciplines (Liu, 2022). In this study, we attempted to address this issue by investigating the effects of live data collection and analysis projects on students' understanding of the role of mathematics in physics and their ability to make sense of real data. Student-driven physics projects incorporated Phyphox smartphone application for data collection (Milner-Bolotin et al., 2021; Staacks et al., 2018). Our research goals were to examine:

- How modern technologies, such as smartphones, can be implemented in a secondary physics classroom to engage students in building meaningful connections between mathematics and science.
- The evolution of student mathematical understanding and capacity to connect mathematics and science, as a result of their project engagement.

Secondary students (N=40) conducted four month-long group projects on the topics of sound that included formulating research questions; setting up experiments that used smartphone-powered data collection; collecting and analysing data; drawing evidence-based conclusions; presenting them to their peers; and producing final reports.

Our initial results indicate that smartphone-based data collection and analysis can help bridge mathematics and science. However, to take advantage of these technologies, mathematics and science teachers should also begin a dialogue and collaboration.

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THE BRIDGING ROLE OF THE CONJECTURE IN THE TRANSITION TO ADVANCED MATHEMATICAL THINKING

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At the university level, a central role in the transition to advanced mathematical thinking (AMTT) is played by problem-solving activities concerning the understanding, the usage of mathematical conceptual entities, and all the processes activating in the generation of examples, up to the formal proof (Tall, 1992). A creative and intuitive field of mathematics that fosters students' active engagement in the discovery and exploration of conceptual entities is topology (Gallagher & Infante, 2022). Creativity manifests itself, especially when managing a conjecture. We face the issue of promoting students' advanced problem-solving competencies and creativity. The research questions are: *What is the role of conjecture in the AMTT? What characterises the strategies of reasoning students use when engaged in proving or disproving a conjecture in topology? How do these factors affect the transition?*

The research was conducted in a third-year introductory algebraic topology course of the bachelor's degree in mathematics. The sample of 14 students, selected among those engaged in all six planned sessions, were assigned tasks to be carried out biweekly, each requiring them to construct examples, counterexamples, and proofs.

We analyse a prove-disprove activity dealing with contractibility and simple connectedness. For any item, we identify to what extent and in what terms the students' proving behaviour shows AMTT signs, looking at the processes activating when constructing a proof or a counterexample: describing-defining; convincing-proving; reproducing-producing, etc. We then focus on the factors triggering them.

The first findings highlight that the *concept usage* in algebraic topology to decide whether certain statements are true or false helps students understand and improve their reasoning. The visual representation is a valuable factor to shift from convincing to proving: the graphical register through which the simple connectedness of the sphere can be intuitively deduced is convincing and preludes to its proving. Furthermore, the protocols highlight as factors fluency, flexibility, and originality. The results are part of an ongoing project about the educational potential of topology.

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TEACHING APPROACHES TO THE DERIVATIVE CONCEPT: USING THE GOAL-ACTION MODEL TO COMPARE THREE TEACHERS' TEACHING APPROACHES

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Derivative is one of the basic concepts in mathematics courses, and understanding it is vital for comprehending other mathematical concepts and their applications. However, many students struggle with understanding this concept, and for many of them, the concept is associated primarily with the technical aspect of the rules of differentiation (Rasmussen et al., 2014). Teachers impact students' learning, and their approaches can affect how students learn mathematical concepts. However, few studies have explored how the derivative concept is taught. This study investigates high-school math teachers' approaches to teaching the derivative. In this paper, we ask: *What are the similarities and differences between teachers' approaches to teaching the derivative, relating to the initial stage of its learning, prior to the presentation of the formal definition of the derivative?*

The study uses the Goal-Action model for Teaching the Concept of the Derivative, based on teaching goals and actions that are required to achieve them (Ayalon et al., in preparation). The model aims to provide practical tools for analysing and comparing teachers' approaches and assessing the learning opportunities for students. The model was developed through semi-structured interviews conducted with high-school leading teachers who teach derivative, followed by combined directed and inductive content analyses. For this paper, the model was used to compare the teaching approaches of three teachers. The study found that teachers use different approaches to teaching the derivative concept prior to the presentation of the formal definition of the derivative. At the conference we will present empirical evidence of these approaches as well as discuss learning opportunities that could be steamed from them. Additionally, we will speculate regarding the perspective of using the model for future research aimed at analysing teaching mathematics.

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PROBLEM POSING AND PROBLEM SOLVING BY PROSPECTIVE TEACHERS: AN ANALYSIS OF THE INFLUENCE OF THE CONTEXT

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Teacher's knowledge of problem solving must include knowledge of problem posing (Chapman, 2015). When a problem-posing activity is undertaken, the context influences the characteristics of the problems posed (Baumanns & Rotts, 2021).

The starting point for this paper is the study by Hartmann et al. (2021). These authors develop an experience in which they present students with real situations that must be used to pose problems that they then have to solve. Based on this experience, and considering the importance of investigating problem solving and posing teachers' knowledge, we will carry out a study with prospective primary school teachers. The aim of this study is to answer the following research questions: RQ1. How does the context influence the characteristics of the problems posed? RQ2. How does the context of the problem influence the problem solving performance?

To address these objectives, we conducted a two-phases empirical study with 106 prospective primary school teachers. First, the participants were given three situations with different characteristics (near-real, far-real and intra-mathematical) from which they had to pose mathematical problems. Then, the participants were then asked to solve three problems, all related to the same mathematical procedure (the Pythagorean theorem) but each formulated in a different context. The results are derived from a mixed analysis. The characteristics of the problems posed and the proposed resolutions have been analysed qualitatively and, subsequently, a quantitative inferential analysis has been carried out to answer the research questions. The results show that prospective teachers are able to pose mathematical problems, particularly from real contexts, although in general the connection with reality is artificial. Regarding problem-solving performance, we found that it is higher for near real-context problems.

Funded by MCIN/AEI/10.13039/501100011033 and ERDF: PID2021-126707NB-I00

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IMPROVING STRATEGIC COMPETENCE IN YOUNG LEARNERS

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This study builds on the positive results of the South African Mental Starters Assessment Project (MSAP) aimed at developing Grade 3 learners' strategic competence which consists of three key skills: fluency, strategic calculation, and strategic thinking (Graven & Venkat, 2021). Strategic calculation and strategic thinking/reasoning are based on a structural understanding of number relations, and fluency with rapidly recalled facts is key for the first two skills (Askew, et al., 2022).

In this study 7 postgraduate students implemented an MSAP unit on jump strategies with Grade 3 classes (132 learners) at 7 schools with different socio-economic status (SES). SES is associated with different levels of mathematical learning outcomes in South Africa. The intervention model consisted of eight 20-minute mental maths activities topped and tailed with similar pre- and post-tests. Part 1 of each test had 20 fluency-related items (2min) and Part 2 had 10 strategic calculation and strategic reasoning items (3min). Our research questions were:

Is there a change in learners' strategic competence after the intervention?

Are these changes patterned on the basis of schools' SES?

School	Sch F	Sch C	Sch D	Sch G	Sch E	Sch B	Sch A
SES	High	High	High	High	Mid	Mid	Low
Part 1	6.8	6.1	4.9	1.3	5.8	4.0	4.6
Part 2	2.8	0.6	1.3	0.4	1.1	0.3	1.3

Table 1. Pre- and post-test mean gains (%) for Part 1 and Part 2 per school

School F (high-SES) had the highest mean gains for fluency (P1) and strategic calculation and reasoning (P2), but no patterned links to SES are evident in the data - akin to the Askew et al. (2022) study. The total mean gains across all schools in the fluency aspect (33.5%) were about 4 times the size of the total mean gains made in strategic calculation and reasoning (7.8%). This result indicates that young learners take longer to develop strategic calculation and reasoning compared to fluency skills.

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COMMUNICATING 21ST-CENTURY COMPETENCIES WHILE BRIDGING BETWEEN CONTEMPORARY AND SCHOOL MATHEMATICS

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Our Oral Communication will address the general issue of communicating OECD (2018) defined 21st-century competencies[1] to high school students via subjects such as mathematics. Our study examines the possibility of introducing these competencies to high school students using Mathematics News Snapshots (MNSs), a set of 27 PowerPoint presentations developed to bridge the gap between contemporary and school mathematics. The theoretical background is rooted in both the need to bridge the gap mentioned above and the need to prepare high school graduates to meet the challenges presented by the changing, unpredictable, and disruptive world of the 21st century. Our study incorporates two stages of which we focus on the first one: Stage 1: Exploring the potential of the 27 MNSs to serve as a vehicle for communicating 21st-century competencies, presumably verifying it. The research question we set for this stage is: To what extent are the 21st-century competencies reflected in each Mathematics News Snapshot? Stage 2: Developing tools for harnessing the potential of the MNSs to serve as a vehicle for introducing high school students to 21st-century competencies and running a clinical implementation study in actual classes. The participants in both stages are high school mathematics teachers. In the first stage, which this Oral Communication focuses on, high school mathematics teachers serve as expert judges. We will present the research tools and preliminary results obtained by twenty high school teachers who served as expert judges and individually mapped, discussed, and reached a consensus on a total of 458 occurrences of 21st-century competencies in a sample MNS known as the Art Gallery Problem[2]. These preliminary results encouraged the ongoing research focusing on the validation of the potential embedded in all 27 MNSs as a vehicle for communicating the 21st century competencies, to be followed by stage 2 which will be an implementation study for applying this potential toward a sustainable solution, as recommended by international bodies like the OECD.

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[1] OECD (2018). The future of education and skills: Education 2030. https://www.oecd.org/education/2030/E2030%20Position%20Paper%20(05.04.2018).pdf

[2] See the full details and the PowerPoint presentation here: https://mns.org.il/the-art-gallery-problem.

LEARNING TO TEACH SIMILARITY IN A LESSON STUDY

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Research shows that the teaching and learning of similarity is not trivial (Seago et al., 2013). This topic was selected by Malawian secondary teachers from two schools who were participating in cycle 2 of Lesson Study (LS), a new practice for them. Huang et al. (2019) report that when LS is introduced in a new context, theory-informed frameworks can scaffold teachers' entry and enable systematic research of teacher learning. A socio-cultural framework used in the LS reported here included attention to learners' engagement with mediational means of how concepts are exemplified and represented, how words are used, and explanations are communicated (Adler & Ronda, 2015). We used content analysis to explore teachers' interpretations of these mediational means in their planning, enactment and reflection sessions on a lesson introducing similar triangles. In this presentation we focus on how teachers navigated the introduction to similar triangles in their lesson, where there were tensions for them on naming (our term) i.e. whether students could discuss activities with similar objects, if the concept had not yet been defined; and representing (our term) i.e. whether everyday objects could be used and not only drawings of triangles. We will show how the introduction to the lesson changed from the first to the second lesson, following reflection and replanning, and that this was a function of simultaneous attention to naming and representing similar triangles, their linkage to each other and to congruent triangles which students had previously encountered. Tensions around naming and representing mathematical ideas are not new. Indeed, they are some of the features of current studies on language responsive teaching (e.g. Prediger, 2022). A practice like LS, we will suggest, opens opportunities for learning to navigate them, in situ.

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PROCESSES OF UTILIZING ACADEMICS MATHEMATICS IN SECONDARY MATHEMATICS TEACHING

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There is substantial empirical evidence that the potential affordances of academics mathematics (AM) for secondary mathematics (SM) teaching are not realized. Various explanations have been proposed as to why this is the case, for example: SM teachers do not see relevance of AM for SM teaching; SM teachers do not sufficiently understand AM to utilize it; utilizing AM in SM teaching is inherently difficult. With respect to the latter, Zazkis (2020) observes that many examples in the literature of utilizations of AM in SM teaching appear to be tacit and personal. These characteristics make utilizations difficult to study, and may explain why so little is known about them.

This study is part of a long-term research project – M-Cubed – that seeks to fill this gap in the literature by exploring processes of utilization of AM in SM teaching in a specially designed 'lab', where mathematicians and SM teachers discuss SM teaching with respect to concrete SM instructional situations (Pinto & Cooper, 2022). This case study explores one particular utilization process. Data for this study consists of: (1) A 1-hour discussion between a mathematician and two experienced SM teachers about a task for 11th-grade students, designed by the teachers; (2) the changes one of the teachers made in the task after the discussion; (3) the teacher's reflection on these changes. Data analysis aimed to elicit tacit and personal aspects the translation process of AM ideas expressed by the mathematician to pedagogical insights. Our findings corroborate the observations of Zazkis (2020). First, by highlighting mathematical claims that were used by the mathematician when reasoning about the task but were not elaborated, and were endorsed by the teacher although he could not fully explain them. Second, by showing that some of the pedagogical insights that guided the teacher when revising the task were not addressed explicitly in the discussion with the mathematician and were only made explicit when the teacher reflected on his revisions. Our findings shed light on *how* processes of utilization of AM in SM teaching are tacit and personal. Further study is needed to refine and elaborate these findings.

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LEARNING TO TEACH VIA MOOC: FROM A DIFFERENT ANGLE – MATHEMATICS TEACHING PRACTICES

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Learning teaching practices is common in teacher education programs (Grossman, 2018). However, prospective teachers do not always have a profound theoretical background to support performing specific teaching practices. In such cases, the prospective teachers often perform the teaching practices ritually. We suggest that the commognitive theory of learning mathematics (Sfard, 2008) as well as the conceptualization afforded by commognition, could provide possible connections to promote teachers' learning of teaching practices.

With these thoughts, we designed a MOOC (massive open online course) for mathematics (in-service and pre-service) teachers and for mathematics educators, that aims at teaching teaching-practices that promote students' explorative participation in the mathematics classroom. The connection between conceptual framework and practice is done by teaching the commognitive conceptualization and theory of learning mathematics as a basis for teaching the teaching-practices. This MOOC opened to the public in January 2022.

The research question we asked is *"how is the commognitive conceptualization that the students learned in the first four units of the course, embedded in the learners' writing about teaching practices in the last four units of the course?"*. The participants in the study are 31 students learning to become secondary school mathematics teachers who learned this MOOC as a mandatory course. The data was taken from 3 sources: forums in which students were asked to write as a part of the course; students' writings in personal blogs that accompanied their learning; and a final questionnaire that they filled in. We analyzed our data thematically.

Findings suggest that the learners related highly to certain aspects of ritual and explorative participation and to the opportunities to learn teachers provide their students.

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MATHEMATICAL MODELLING INSTRUCTION BY PRE-SERVICE TEACHERS

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Mathematical modelling (MM) has become increasingly prevalent in mathematics education. Yet, studies highlight challenges in instructing MM in class (Kaiser, 2020), thus raising the need to support teachers in their training for such instruction (Blum, 2015). Therefore, identifying best-instructional practices (BIP) that encourage MM implementation is crucial. MM involves a cyclical process in which a real-world situation is translated into a real-world model which is then solved using mathematical principles and interpreted back in the real-world situation (Kaiser, 2020). MM tasks have the potential to invoke BIP as defined by the NCTM (2014), as these practices reflect abilities that are aligned with the MM competencies (Kaiser, 2020). Accordingly, this study aims to explore the degree to which pre-service teachers use and manifest BIP during MM instruction comparing to standard math lessons.

Participants are nineteen pre-service teachers in their final year of professional training. First, their BIP were assessed at the beginning of the training in a school setting, in order to monitor their ability to implement BIP based on the theoretical knowledge they acquired during their studies. Further their BIP were assessed towards the end of the training in a simulation lab while instructing MM tasks and in a school setting. A rubric designed for this study was used to capture observable events for each BIP.

One-Way ANOVA with Repeated Measures revealed that pre-service teachers' implementation of BIP differed significantly between all three measurements, $F(2, 6) = 24.59$, $p < .001$, $\eta 2 = .89$. A follow-up Bonferroni test indicated significantly higher BIP implementation in the laboratory (M=25, SD=2.5) and on the second measurement in class setting (M=18.5, SD=5.1) than the first measurement in classroom (M=6.8, SD=1.0). The effect size between the first and second measurements was high (d=0.9), but low between the first and third measurements (d=0.3). Difference between laboratory and second classroom measurement did not reach significance. Findings indicate that pre-service teachers' levels of BIP implementation were positively impacted by experimenting in the simulation lab while instructing MM tasks.

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A DESIGN BASED STUDY: GAME MATERIAL DEVELOPMENT EXPERIENCES OF MATHEMATICS TEACHER CANDIDATES

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Games are as old as human history (Miller, 2008). Contrary to the idea that play is the opposite of work or seems only childish, it makes learning fun and requires intense study and knowledge (Oblinger, 2006). Despite its importance, it is difficult to say that the game is used in mathematics education as an effective teaching tool today. Beyhan and Tural (2007) states some reasons why teachers do not include the games in their learning process; inadequacy in the knowledge of the game as a teaching method, limited game materials and difficulty in designing/providing an appropriate game material. Thus, the awareness of teacher candidates, who will carry out the teaching profession with this awareness, is important to be developed and searched. In this study, mathematics teacher candidates' experiences with a design-based study process, which enables the design, development and evaluation of educational products (Plomp, 2013), were examined. The data were collected from 37 pre-service mathematics teachers who were involved in the 4-phase of this design-based process: preliminary research and prototype-1 development, prototype-2 development, prototype-3 development, final game material development-field application (at elementary schools). The qualitative data obtained through observations, instruction forms, classroom presentations, and prototype reflection papers were analyzed by following the steps expressed by Miles and Huberman (1994). The findings of the study were gathered under 3 categories as teachers' experiences in design process, in classroom application process and their experiences about group work. As the common products of design-based research; game materials which were nourished by theoretical and practical infrastructure and professional development of teachers have been obtained.

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IMPLEMENTATION OF INSTRUCTIONAL DESIGN ARRANGED WITH AUGMENTED REALITY IN TEACHING THE SUBJECT OF 6TH GRADE GEOMETRIC OBJECTS

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It is more difficult for students to try to make sense of abstract concepts, such as geometric objects, in their minds than concrete concepts. We can use computer-generated virtual environments and augmented reality applications to gain abstract concepts (Estapa ve Nadolny, 2015). In this study, we developed an augmented reality application that will be used in a lesson on "Geometric Objects" in mathematics for sixth graders. Additionally, it aims to investigate the impact of instructional design created with the aid of this application on students' conceptualization of three-dimensional objects. For this reason, a design-based research method was chosen as the research methodology.

We selected the study group of the research as eight students from the three classes that the researcher attended, among the 6th-grade students in Eskişehir. The research employed both qualitative and quantitative data collection methods. The qualitative data tools of the research consist of expert opinions, students' activity papers and rubrics, video recordings, and student interview questions. Quantitative data tools were applied in the form of pretest-posttest in the research; "Building and Drawing Geometric Structures Achievement Test", "Spatial Ability Test" and "Augmented Reality Applications Attitude Scale".

The research results showed that the students made different inferences by having experience with the activity papers and had no problems generalizing the volume.Considering the average scores on the achievement test of building and drawing geometrical structures, the students scored higher on the final test. When the spatial ability test averages were examined, it was seen that the post-test average was higher. The average score of satisfaction using augmented reality applications is higher in the posttest. In addition, the average score of students' anxiety about using the application decreased. When the students' views on the usefulness and educational effect of geometry teaching using the augmented reality application were examined, there were statements that the application was easy to use, and their positive attitudes increased.

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AN EVALUATION OF STATISTICS LESSONS AT SECONDARY SCHOOL WITH REGARD TO GAISE REPORT-II.

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The present study is based on the research carried out within the scope of project number 220K338 supported by The Scientific and Technological Research Council of Turkey (TUBITAK). The GAISE Report II (2020) recommends that students, who can easily access data as a result of the digital revolution, need to be educated more comprehensively in order to evaluate and manage this data critically, objectively, and logically. This fact led us to search how our current statistics teaching in schools can be strengthened based on the GAISE recommendations. Thus, our project has emerged from this urgent need. The project consists of two parts. The first part of the project is to clarify to what extent the secondary school teachers' lessons reflect GAISE's recommendations. The second is to develop a teacher guide that focuses on GAISE's recommendations.

In the first part of the project, participants were six secondary school mathematics teachers. Their lessons on statistics topics were observed, recorded, and transcribed. Data was analysed qualitatively. Content analysis is used to determine themes. Findings were presented through these themes.

The preliminary findings of the first part of the project illustrated that the teachers' statistics lessons were far away from the GAISE recommendations. Teachers preferred to build their lessons on procedural understanding. They asked knowledge-level questions that only required procedural steps instead of conceptual understanding, and gave some limited examples from real-life situations that were not sufficient to create statistical context. Therefore, the statistical process as a whole could not be observed throughout their lessons as well.

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FUNCTIONS AND THEIR GRAPHS IN HIGH SCHOOL MATHEMATICS IN NEPAL

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There is an accepted notion within mathematics communities that functions and their graphs are one of the most important areas of mathematics. The introduction of algebraic and graphical representations of functions can be seen as one of the critical moments in mathematics learning and represents 'one of the earliest points in mathematics at which students use one symbolic system to expand and understand another' (Leinhardt et al., 1990, p.2).

However, in the context of Nepal, there are issues in the learning process of basic functions and their graphs (polynomial, trigonometric, exponential, logarithmic, and rational functions) in high school mathematics (grades 9-12) pertaining to the ability to produce an outcome that includes the necessary skill for the graphing of these functions and applying them in various areas such as calculus, mathematical modeling, differential equations, and vectors. I conducted a comprehensive examination of the curricula exams of grades 9-12, and determined that the following are the most obvious issues that could be hindering the attainment of the desired learning outcomes.

Issues in the curriculum/Tests

- Lack of clarity in the learning objectives and outcomes
- Insufficient weight being placed in general
- Premature introduction of "Limit and Continuity" in grades 9 and 10
- Lack of well-formulated questions on tests
- Avoidance of inclusion of applications of graphs on tests
- Very predictable exam pattern

The outcomes of my analysis of the curricula and tests suggest that the syllabi do not properly incorporate the graphing of functions and their applications, and the testing system fails to test the learners effectively in maximizing the learning outcomes.

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INTERROGATING STUDENTS' RESPONSES TO ASSESSMENT TASKS: A FOCUS ON THE FRACTION TWO-THIRDS

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Researchers have highlighted the importance of assessment task design to identify students' mathematical knowledge, skills, and understandings (see for example, Kieran, 2019). This paper discusses the results for over 600 middle-years students (ages 10 – 16 years) for two fraction tasks (Figure 1) from two paper-and-pencil tests (Pearn, 2019). Both tasks include the quantity representing two-thirds and students are asked to find the quantity representing the whole and write detailed solutions for both tasks. Task A uses symbols while Task B is a worded task with a diagram.

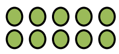
Task A: Write a number in the box to make a true statement. $\frac{2}{3} \times \square = 18$	Task B: This collection of 10 counters is $^{2}/_{3}$ of the number of counters I started with.  How many counters did I start with?

Figure 1: The two written fraction tasks for two-thirds

Fraction tasks are usually presented in textbooks as symbolic equations which require middle-years students to find a fractional part of a whole. In order to solve Task A students must interpret the equation and then find the quantity representing the whole given two-thirds of that whole. While Task B also requires students to find the quantity representing the whole the interpretation has been provided in the worded task and the accompanying diagram. An analysis of the results for these two tasks highlighted a large difference in the percentage of correct responses. Students were more successful with Task B than Task A. While 77% Year 5, 81% Year 6, 88% Year 8 and 93% Year 9 students gave correct responses for Task B only 13% Year 5, 20% Year 6, 30% Year 7 and 87% Year 9 students gave correct responses for Task A. Students' strategies for Task B ranged from explicit partitioning of diagrams before using additive strategies to fully multiplicative strategies including the correct use of algebraic notation.

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DEVELOPMENT OF UNIVERSITY STUDENTS' MATHEMATICAL BELIEFS IN INTERDISCIPLINARY MATHEMATICS CURRICULUM

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Research on the development of mathematical beliefs suggests that beliefs may change----especially in long and experience-rich circumstances (Safrudiannur, Bekje & Rott, 2022). However, this line of research has mainly focused on students/teachers who were from the discipline of mathematics and their experiences in either pure mathematics or mathematics education courses. Also, it lacks enough insights from students from other disciplines and with experience in interdisciplinary courses contexts. This study addresses these research gaps by exploring belief changes of university students from various disciplines in interdisciplinary mathematics curriculum. And the research questions are: Do university students' beliefs about the nature of mathematics differ before and after attending an interdisciplinary course? And (if so) what curricular factors contribute to students' change in beliefs?

The study included a total of 216 university students (female: 46%, male: 54%; natural sciences: 72%, social sciences: 28%) enrolled in an interdisciplinary course *Math, Science and Arts*. They completed a pre-post survey consisting of demographic information and an open question on their views of mathematics before and after the course. The data was categorized in terms of four mathematics-related orientations: Formalism, Scheme, Process, and Application, and further quantified for the McNemar-Bowker test. 10 students' final text on their reflections on the course were included and coded. The interrater agreements of the codes were acceptable.

Results show dramatics changes in students' mathematical beliefs: there is a significant difference for social science students' belief before and after the course ($\chi^2(6)$ =16.014, p=0.014); students' beliefs in application were developed from "mathematics helps to solve daily problems" to "explains how everything works", formalism developed from "mathematics as a science of numbers" to "patterns". Next, interdisciplinary contents centred on the mathematical topics, learning-by-doing pedagogical method, and interdisciplinary project were identified as the main factors leading to students' enhanced understanding of the nature of mathematics. The study suggests that the mathematical beliefs of students from various disciplines can be changed and it also provides implications for the teaching of interdisciplinary mathematics curriculum.

Acknowledgement The work was conducted under the grant (2022JY056).

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AN ANALYSIS OF VIDEOS PRODUCED DURING THE PANDEMIC FROM MAYER'S MULTIMEDIA LEARNING PERSPECTIVE

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1. INTRODUCTION, PURPOSE AND TEORETICAL FRAMEWORK

The social isolation caused by the COVID-19 pandemic has demanded, in the educational field, the development of teaching-learning strategies such as video production. In this context, mathematics teachers from a Brazilian educational institution, CEFET-MG, have produced and made available videos on YouTube. Our aim is to present an analysis of these videos from the perspective of Multimedia Learning Principles presented by Mayer (2009) who cites 5 cognitive processes for meaningful learning, and these involve selecting and organizing what is relevant in words and images and integrating these representations.

2. METHODOLOGY AND DATA ANALYSIS

CEFET-MG offers professional high school courses in 9 cities in Minas Gerais State, serving about 6,500 students. In the 1st grade, seven mathematics contents are taught, which led the teachers to produce 87 videos with an average duration of 15 minutes each. We counted the accesses to each of them in 2022, used the data to make graphs, and identified a similar behavior in the variation of accesses to the videos of each content, being the 1st video the most accessed and the last one, the least. We noted that there are different strategies in the presentation and organization of what is explained in each video. Only in the Trigonometry content the most accessed video was among the last on the list, and our analyses allowed us to identify that the most accessed ones present, in a more evident way, elements that facilitate the selection and organization of information, allowing the perception of what is most relevant, and the ones with little visualization have more focus only on a mathematical notation presentation and the respective teacher's explanation.

3. RESULTS AND FINAL CONSIDERATIONS

We understand that there may be other explanations for this variation in the number of views in each video, but we identified greater access to those that enable greater management of cognitive processes, indicating that videos that follow these principles have a greater chance of success.

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EXPANDING THE SINE CONCEPT FROM A RATIO IN A RIGHT-ANGLE TRIANGLE TO A CYCLIC FUNCTION: THE CASE OF ANGLES GREATER THAN 360^0

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Sine function may be defined as a ratio between sides of a right-angle triangle, or it may be defined through the unit circle. Students who firstly learn the definition of sine in a right-angle triangle may encounter a conflict when the extended definition of sine is learned and the “rules of the game” change. The conflict may arise between the previous narratives and the new narratives derived from the extended definition (Sfard, 2007). Together with the expansion of the sine definition, the concept of angle also needs to be expanded, from a positive value that is bounded by 360^0 to any value.

This research aimed to explore the process of expanding students discourse about the concept of sine from its definition as a ratio in a right-angle triangle to its expanded definition in the unit circle, among high School students, working on purposefully designed research activity in a technological environment (Perlmutter, 2022).

Four pairs of students participated in the study. The activities were video recorded and transcribed verbatim, and together with the filled activity sheets and saved application files formed the data for the study. The data was analysed based on the commognitive approach (Sfard, 2007). The student's narrative changes regarding the object Sine and related objects were examined. In this report we focus on the case of angle size.

The findings show that all participants changed their discourse regarding angle sizes. All participants initially defined angles as a positive number, smaller than 180^0 or 360^0. Although some managed to draw angles larger than 360^0, they insisted it doesn’t exist. Through the designed activities all participants successfully expanded the angle concept beyond 360^0 to infinity in the positive direction. Also, two students hinted in their discourse to possible existence of negative angle. The results might help teachers address pivotal points of difficulties while teaching trigonometric functions in class. In the presentation, quotes and visual mediators drawn by students will be presented.

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STUDENTS ATTITUDES TOWARDS METACOGNITIVE SKILLS FOR STRATEGIC MATH PROBLEMS

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Metacognition is an important aspect of problem-solving. By being aware of metacognitive skills, individuals can improve their performance through effective planning and monitoring throughout the problem-solving process. Improving metacognitive thinking requires strengthening learning strategies that include identifying the problem, creating a solution plan, monitoring progress, and evaluating the solution. This study examines the effect of explicit and implicit teaching methods on students' attitudes toward metacognitive skills while solving math problems using mathematical strategies such as recursion and proof by contradiction. The study was conducted within the framework of "Kidumatica" – an after school mathematics enrichment program for mathematical promising students. A total of 81 students in 6th-7th grade participated in a six months program, designed to teach mathematical problem solving strategies. 38 students studied with the explicit teaching (ET) method and 43 students studied with the implicit teaching (IT) method. Data was collected at the beginning and end of the program using two metacognitive inventories: HISP, which assesses students' thinking while solving mathematical problems, and Jr. MAI, which evaluates students' metacognitive thinking and ability to control and reflect on the learning process. Our findings showed that in both inventories, mathematical promising students showed positive attitudes towards metacognitive knowledge and skills, with no significant difference between the two teaching methods. At the end of the program, HISP results showed a significant improvement in planning and monitoring skills in the ET group compared to IT group ($t(72) = 2.37$, $p < 0.05$, ET group 3.92 ± 0.44, IT group 3.63 ± 0.58). The Jr. MAI also indicated that the ET group had a better attitude towards metacognitive skills (mean = 4.16 ± 0.45), while the IT group had no significant change (3.79 ± 0.79).

In conclusion, our study found that explicit teaching was effective in improving students' attitudes towards metacognitive skills, particularly in planning and monitoring. The use of explicit teaching to impart mathematical strategies enabled students to better understand their thinking process and encouraged positive attitudes towards problem solving (Dori, Mevarech, & Baker, 2018). These findings have important implications for mathematics education and the role of explicit teaching in developing metacognitive skills.

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WHAT'S YOUR PROBLEM? – ADAPTING DEWEY TO INFORM MATHEMATICAL INSTRUCTION IN HIGH SCHOOL

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Topics such as financial products and access to financing opportunities – but also other topics drawing interest due to societal change like climate change and new technology – got of increasing relevance for coordinating every-day life. Mathematics plays a crucial role in becoming able to responsibly navigate these topics and related issues. In order to integrate this change in the mathematics classroom and linked to other disciplines, it requires a didactic approach that operates closely with society, their citizens and – especially – the lived world of the students. Drawing on an approach to education informed by Dewey's pragmatism (Dewey & Hinchey, 2018) allows to involve the students' view on their lived world in laying out instructional design, seeking to avoid the risk of them not perceiving such topics as affecting their life. This in mind, I explore the question '*What are current problems of broader interest for students that connect to mathematical content and meaning approached in high school mathematics – and how can connections meaningful to students can be initiated and developed across grades?*'. This preliminary study sets the ground for the development of a sequence of problem-based sessions across school grades to foster the development of mathematics as meaningful to students – in my case especially financial literacy, constituting the first phase of a larger design-based research project.

To gain a tentative answer to the question of the preliminary study presented here, I carry out narrative interviews with 10 students aged 15-18. This method of data collection in form of storytelling is suitable to discover interpersonal and intrapersonal experiences through narration of past significant situations and reflecting on them (Saldaña, 2021). Through qualitative content analysis and subsequent comparing and contrasting cases of problem situations across students, I seek to identify recurring problem situations relevant to the learners' lived experience and lived-in world.

In my presentation, I will describe these different problem situations and suggest potential links to fundamental ideas in (high school) mathematics. I will furthermore give an outlook on how these results will be integrated for designing the teaching sequence sought to foster financial literacy through mathematical education.

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INTEGRATING COMPUTATIONAL THINKING WITH ABSTRACTION IN CONTEXT: A PROJECT PROPOSAL

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The concepts of Computational Thinking (CT) and its link to mathematics provide an assumption that the affordance of CT can be extended to mathematics (Wing, 2006). Correspondingly, their interplay suggests a potential approach in helping the learners abstract concepts that are new to them. In this light, the researcher will present her proposed dissertation project that aims to fill in the gaps concerning CT and mathematics education. The proposed research intends to find out the CT skills and practices that the students will manifest as they perform the unplugged CT task involving the addition of rational algebraic expressions (RAE). Moreover, the research will also describe how CT, through the unplugged Use-Modify-Create (UMC) progression (Lee et al., 2011), mediates in vertically reorganizing the students' previous constructs into forming the concepts about adding RAEs.

Data collection will commence in September 2023 in-class in which three triads of junior high school students, ages 13-15, will perform the CT-integrated tasks. The researcher will examine the written outputs and scratch pads of the students, review the recordings, and backread through the transcripts. The data analysis will utilize the Recognizing, Building-with, Constructing (RBC) model of Abstraction in Context (AiC) to pinpoint the mathematical concepts, and identify the different actions and utterances of the students leading to the construction of the knowledge about adding RAEs (Hershkowitz et al., 2020).

The researcher hypothesizes that the CT-integrated mathematical task is a context that incites students to reorganize prior mathematical constructs in forming the new abstract construct. Such results are vital in generating educational interventions for deepening understanding, correcting misconceptions, and conducting curricular reforms related to teaching and learning RAE.

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PST'S PERCEPTIONS OF TOPIC DIFFICULTY: PD PREDICTOR OF PST'S

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South Africa has a dearth of quality Mathematics teachers. PGCE students aspire "to excel in their skills of Mathematics teaching". It is therefore imperative to predict their level of PD to pitch on how to teach them. PD is linked to the quality of teaching, which is determined by the TSTK possessed. This paper reports on the perceptions of PGCE student-teachers on the difficulty of Mathematics topics taught in grade 12 in South Africa. Pre-service teachers were given prompt questions to assess the level of difficulty of the topics taught in grade 12 Mathematics on the 5-point scale. The gist of the matter being the reasoning behind their rating of the level of difficulty of the topics.

This study was underpinned by the construct of **professional noticing** coupled with the **T-STAF** (Ramabulana & Sedumedi, 2017)**.** If pre-service teachers have accurate perceptions of the learners' difficulty, they are in a better position to assist learners to overcome their difficulties (Jacobs, Lamb, & Philip, 2010; Alwast & Vorholter, 2022). Thus, professional vision or intentional noticing is one of the components of teacher's competence, especially when they can identify or predict significant and noteworthy aspects of the topic to learn.

This is an on-going qualitative study, where data was collected from the PGCE class of Mathematics Education at a university in South Africa. Pre-service teachers were given a task to rate the level of difficulty, and justify their rating based on the knowledge they have about the topic. Analysis on pre-service teachers' perceptions of the levels of difficulty against the educational ends of the topics revealed a low level of PD, which gives an indication of the direction of tuition in the course to impact of their skills of teaching Mathematics at FET.

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ENACTIVISM AND REFLEXIVITY: SOME CONSIDERATIONS IN RESEARCH ON MATHEMATICS EDUCATION

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According to the Encyclopedia of Qualitative Research Methods (2008, p. 748),

> Reflexivity can be broadly described as qualitative researchers' engagement of continuous examination and explanation of how they have influenced a research project

Although some methodological questions are present in mathematical qualitative education research, such as whether these methods address the research question by the authors? What is the theoretical standpoint concerning the research? Explicitly, each author's reflexivity process is less known. In addition, the structure of research articles in mathematics education is narrowed to an ideal-typical journal paper characterised as an introductory section, theoretical framework, next comes methodology (including methods), and next section, typically called 'results' or 'findings' (Niss, 2018, p. 42) How do we describe what we have learnt on the learning of others? What happens with the unconscious vias observing mathematics? Is reflexivity a point to consider in reports about mathematics education research?

In this Oral communication, based on the Enactivism theory, which recognises the fundamental circularity between the researcher and the research (Varela et al., 1993), I present how Enactivism supports my reflexivity, the interplay between research and the participants, noting that in this interaction, I, as an observer/researcher, am also learning in this process, seeing my influences on the study which I named: avoiding mathematical judgments beforehand and, reflecting on coherent mathematical behaviour in the observations. The study took place in a mathematics grade-eight classroom (ages 13–14 years old students), doing methodological research about re-observing the emergences of mathematics learning. Finally, I discuss further research is needed on the process of reflexivity in mathematics education and its possible implications doing research.

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HYBRID TEACHING IN MATHEMATICS CLASSES: STUDENTS' PERSPECTIVE

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The Israeli education system shut down in March 2020 due to the spread of COVID-19 and switched to a hybrid teaching format. This study examines 11th grade students' attitudes towards mathematics in the period of hybrid teaching during the COVID-19 pandemic, and how (if at all) these attitudes differ, from the students' point of view that they held before the pandemic, in relation to three dimensions of Di Martino and Zan (2010): (1) their perceptions of mathematics, its learning and teaching, (2) their perceptions of self-efficacy in mathematics, and (3) their emotional tendency towards mathematics. The research question: *What are the attitudes of students studying math in the eleventh grade at the 5-unit level about hybrid teaching in mathematics during the COVID-19 pandemic compared to their attitudes towards "normative" classroom teaching as they experienced before the pandemic?* Attitudes are examined regarding the three dimensions: mathematics, its learning and teaching, self-efficacy in mathematics, and emotional orientation towards mathematics.

Semi-structured individual interviews were held with 24 students who studied math in hybrid manner. Combined direct and inductive analyse revealed: (I) Three profiles of students in relation to changes in their attitudes towards mathematics and its learning during the hybrid teaching in mathematics; (II) For all 24 students, the hybrid teaching aroused their responsibility for learning mathematics and their self-reliance in learning mathematics. In terms of the contribution to the field, the research findings point to the advantages and disadvantages that the students experienced in the hybrid teaching of mathematics. Although, at this stage, the findings are limited to a small number of students, nevertheless, the feasibility of thinking about a combination of face-to-face teaching with remote teaching in the routine teaching of mathematics and not only in emergency situations is apparent. Also, the categories identified and the profiles built may serve as a basis for discussion with teachers and teacher educators about hybrid teaching that can provide a response to different students. At the conference, we will present the way of analysing the data and the findings that show the students' attitudes towards the hybrid teaching of mathematics during the Corona pandemic, which they experienced compared to what was before.

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TOWARDS A COGNITIVE PROCESS FRAMEWORK ON THE LEARNING MECHANISMS OF DIGITAL TOOLS IN MATHEMATICS

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INTRODUCTION

In order to explain successful mathematics learning with digital tools, the specification of learning activities is crucial for any empirical investigation. We introduce a cognitive process framework for the mechanisms of (mathematics) learning with digital tools that combines core ideas from the psychology of instruction (utilization-of-learning-opportunities models; ULO; Seidel & Shavelson, 2007), cognitive psychology (Knowledge-Learning-Instruction framework, KLI; Koedinger et al., 2012), and mathematics education (content-specific theoretical & empirical analysis).

Our framework highlights the mediating role of specific cognitive processes in the cause-and-effect mechanisms of successful learning with digital tools. That is, the digital tool as central predictor includes the design of the environment as well as the inclusion of instructional features. We argue that to find the appropriate to-be-implemented features, knowledge about content-specific learner models (i.e., learning processes and knowledge components) is of use to design instructional events that are well-suited to stimulate the relevant generative cognitive processes.

We demonstrate how this framework can be used to theoretically ground the evaluation of students' use of such digital tools based on log-files of unobtrusively measured student-tool interactions: We describe how to investigate these processes *during* learning by linking students' (external) on-task behavior and their (internal) cognitive processes (Hahnel et al., 2019), so that a causal interpretation on learning effects is possible.

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IMPROVING MATHEMATICS TEACHER EDUCATORS' PRACTICE THROUGH COLLEGIAL OBSERVATION AND JOINT REFLECTION

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Mathematics teacher educators (MTEs) at an institute for initial teacher education (ITE) in primary mathematics formed a community of practice (Wenger et al., 2011), defined their own values, and engaged in self-initiated and self-organized professional development. The MTEs come from diverse professional backgrounds and experiences, including former primary, lower or upper secondary school teachers or researchers in different fields of mathematics education. As there is no formal training path to become an MTE, each member has gone through an individualized professional transition process (Beswick & Chapman, 2020). Initially, small groups of MTEs formed ad hoc to plan seminars and share experiences. Then the entire collegium became involved. The first step was to agree on a set of shared values by addressing the questions "What motivates us?" and "What identity do we aim to establish?" Subsequently, the MTEs met regularly over a period of about a year to discuss criteria for good teaching in ITE, based on theoretical approaches and practical experiences from the different areas of expertise of the MTEs. The agreed criteria served as the starting point for the development of a peer observation tool to be used for collegial supervision, i.e. observation of lectures and seminars followed by joint reflection.

In the course of the summer semester 2023, the observation tool will be regularly used in the collegium and its suitability will be investigated. The research question is: To what extent is the peer observation tool in an individual collegial supervision suitable for achieving concrete development goals of an MTE? A mixed methods design will be used to gather data, which will consist of observation protocols, transcripts of the ensuing joint reflection, and written self-reflections by the observed MTEs after a few weeks, in which they recall their individual learning process and account for changes in teaching as a result of the supervision. These documents will be analyzed using qualitative content analysis. At the PME conference, the observation tool, the process of its development and the first research results will be presented.

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STUDENTS WITH LEARNING DISABILITIES EXPRESSING MATHEMATICAL CREATIVITY

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One of the ways to strengthen the connections between different mathematics topics and expand existing knowledge is to encourage mathematical creativity (MC). MC is usually assessed by fluency (the number of final solutions), flexibility (the variety of solution strategies or approaches) and originality (uniqueness of the solutions). It is well known that multiple-solution tasks (MSTs) can stimulate divergent thinking and invite students to create a variety of paths leading to the same solution, thus encouraging creativity (Levav-Waynberg & Leikin, 2012). This study focused on students with learning disabilities (LD) and their engagement with MSTs. The research questions are: To what extent do students with LD demonstrate fluency and flexibility when engaging in MSTs?

The study included 12 third grade students with LD from two special education classes located within general, mainstream schools. The students worked in pairs on the following MST "How many arithmetic sentences can you create that have 12 as the solution?". The researcher carried out mediational interactions based on Feuerstein & Feuerstein (1991). Solutions were coded according for fluency and flexibility.

Fluency findings showed that 96 arithmetic sentences were written, 88% of which included one type of operation. The most common operations were addition (46%) and subtraction (30%). Only 10% of the sentences included multiple operations (indicative of flexible thinking), with three or more numbers (e.g., 5+7–6+6). Students used an average of three operations, created sentences with three numbers or more (also a sign of flexibility), and used a variety of multi-digit numbers (e.g., 21–9, 112–100, 4012–4000).

Results indicate that students with LD can express fluency and flexibility when they engage in MSTs. Providing mediational interactions can play a part in making MSTs accessible and suitable for students with LD.

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LESSON PLAYS TO ENVISION DISCUSSION FACILITATION

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Facilitating mathematical discussions is essential for promoting students' problem-solving abilities (e.g., Nathan & Knuth, 2003). And so, teacher preparation programs teach pre-service teachers (PTs) to facilitate mathematical discussions around problem-solving tasks. Moreover, through teaching PTs to facilitate mathematical discussions around tasks, teacher education programs strive to construct PTs' vision to follow reformed perspectives of mathematics teaching. Yet, little is known about how PTs envision themselves enacting this practice in their future classes. Thus, I propose to get a glimpse into PTs' envision of facilitating mathematical discussions using the construct of *lesson play* together with Nathan and Knuth's (2003) notion of *analytic* and *social scaffolding*. A lesson play is a script of a hypothetical dialogue describing how a lesson or a part of a lesson may unfold (Zazkis, 2017). Analytic scaffolding is the support provided to students so they will build their mathematical understanding of ideas and concepts. Social scaffolding is the support that generates students' engagement in the discussion. I ask: *How do PTs envision themselves facilitating future mathematical discussions around a task in terms of analytic scaffolding, social scaffolding and information flow, as manifested in a lesson play they wrote?*

Data include lesson plays written by seven PTs enrolled in an academic course. The PTs chose an algebra task from a frequently used textbook and wrote their lesson play around it. Data analysis included top-down and bottom-up categorizations and revealed that six out of the seven PTs' lesson plays are characterized by analytic scaffolding, while only one of the PTs' lesson plays is characterized by both analytic and social scaffolding. Further, five of these lesson plays included teacher-to-class/student or student-to-teacher talk, whereas two PTs also included student-to-student talk. From these results, I draw two patterns of PTs' envisioning of mathematical discussions. In the presentation, I will demonstrate these patterns and align them with other research findings. I suggest that using these patterns can support teacher educators when working with PTs, to address PTs' envisioning early in the teacher education program.

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AGENCY IN THE EXPLORATIVE PEDAGOGICAL DISCOURSE WITHIN A MINORITIZED TEACHERS PROFESSIONAL LEARNING COMMUNITY

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In the TEAMS project – Teaching Exploratively for All Mathematics Students, we introduce teachers to practices that afford students opportunities to struggle and promote students' authority and conceptual understanding. These practices are valued within the Explorative Pedagogical Discourse (EPD). Within a discursive approach to teachers' learning, when teachers adopt new teaching practices they become participants in a new Pedagogical Discourse while individualizing this new PD to be their own discourse (Heyd-Metzuyanim & Shabtay, 2019). Yet the EPD may not align with the pedagogical discourses of teachers in minoritized communities. The Israeli project is built around Professional Learning Communities (PLC), part of which are for Arab teachers, led by teacher-leaders who themselves are newcomers to the EPD. Participating in these PLCs offers the teacher-leaders and teachers' opportunities to author identities and enact agency while becoming part of a community of practice (CoP). Our study aims to understand how PLC participants enacted agency with relation to the EPD. We define agency as moments of discourse where teachers expressed and asserted their beliefs, values, and knowledge to make choices in the given context (Barton & Tan, 2010). Using video-recordings, we focused on teacher-leaders and participating teachers' discourse around slides handed to them from the PLC leaders. We analysed teacher-leaders' choices of how to present these slides, and the agency afforded to the participating teachers. We found that PLC sessions can be differentiated concerning participating agentively in a CoP and adopting the EPD. We present two contrasting sessions. In one the teacher-leaders presented the slides in ways that were aligned with the EPD yet there was minimal teacher participation in the discussion. In the second, the teacher-leaders deviated from the EPD and the intended goals of the slides, yet this choice offered teachers agency to reflect on various pedagogical practices of teaching Arab learners. We conclude that there may be two orthogonal axes at play: participating agentively in a CoP and adopting the EPD. We discuss the meaning of PLC discussions being high or low on these two axes.

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REPRESENTATIONS IN PROBLEM-SOLVING PROCESSES WITHIN AN ENRICHMENT PROGRAM FOR PRIMARY SCHOOL CHILDREN

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Hoffmann (2004) highlights the importance of transformations in constructing diagrams for the process of gaining new mathematical insights within a semiotic-pragmatic framework. He explains that playing with transformations within experimenting upon icons or diagrams is crucial for making the necessary associations to identify ideas and potential solutions. The ability to transform representations is thus crucial for successful mathematical problem solving. This ability is a characteristic of mathematically gifted children (Assmus & Fritzlar, 2018) who show strong competencies in solving mathematical problems. Therefore, it should be possible to analyse transformation processes in a setting with mathematically gifted children. Therefore, a study was planned guided by the following research question: "What processes of transformations of external representations do gifted children use in their problem-solving process?"

The study is associated with an enrichment program for children, aged 8 to 14, who have high interest in mathematics. The collected data consists of 13 video recordings and its transcriptions of interviews with 18 children. The transcripts were analysed using qualitative content analysis (Mayring, 2008), and the coding categories were developed deductively. The categories were divided into the perspectives "diagrammatization" and "new mathematical insights" to identify transformation processes connected to new mathematical insights.

The results indicate that Hoffmann's concept of transformations in constructing one's own diagrams to gain new mathematical insights can also be applied to children. An exemplary case will be presented. Future research has to show if and how these transformation processes can be stimulated to support new mathematical insights in problem-solving processes.

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EFFECTS OF SHORT INSTRUCTIONS ON SOLVING OPEN MODELLING PROBLEMS

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In everyday life, humans face open modelling problems that do not include all the essential information and require making assumptions. Solving open modelling problems requires identifying missing quantities and making numerical assumptions about unknown quantities. Analyses of students' solution processes (Galbraith & Stillman, 2006; Krawitz et al., 2022) indicated that both activities might be challenging for students. In order to analyze how to support students, we set up a randomized control trial. The hypotheses of the study were: (1) Instructional prompts on what quantity is unknown will positively affect modelling compared to no instructional prompts. (2) Instructional prompts on what quantity is missing and information about what assumptions should be made will positively affect modelling compared to instructional prompts on what quantity is missing only.

Three hundred sixty-five ninth, 10th, and 11th graders (49% female, mean age 15) were randomly assigned to an unknown quantity group (UQG), assumption group (AG) or no instruction group (CG). Instructions were "To solve the problem, you must estimate …" (unknown quantity in the problem such as the diameter of the speaker in the problem Speaker in Krawitz et al., 2022) in UQG and "To solve the problem, you must know … (unknown quantity, such as the diameter of the speaker). Assume that the diameter of the speaker is 8 cm." in AG. Students worked on six problems (0 = no or false solution; 1 = accurate solution; Cronbach's α for modelling test = .78).

The results showed that students in the UQG outperformed students in the CG ($t(129.47) = 7.15$, $p < .001$, Cohen's $d = 0.88$) and that students in the AG outperformed students in the UQG, $t(226.37) = 3.83$, $p < .001$, Cohen's $d = 0.51$. This study confirmed that identifying missing quantities and making numerical assumptions about the missing quantities are important barriers to solve modelling problems. Furthermore, the results indicate that short instructional prompts can be used by teachers to support students in the classroom.

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ASSESING THE DIAGNOSTIC COMPETENCE OF IN-SERVIC TEACHERS DURING MODELLING INSTRUCTION

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Mathematical modelling refers to the process of solving authentic real-world situations by applying mathematical instruments, and is defined as a cyclic process consisting of several phases: *understand simplify*, *mathematise*, *mathematical work*, and *interpret and validate*. One critical dimension of teachers' professional competence for modelling instruction is the diagnostic competence, which refers to the ability to identify the challenges students encounter during the modelling process (Borromeo Ferri & Blum, 2010). Studies in this field have focused on exploring the diagnostic competence of pre-service teachers or challenges faced by students, leaving a gap in research on the diagnostic competence of in-service teachers during real-time practice. Considering prospective, objective, and retrospective views, this study explores in-service teachers' diagnostic competence when enacting modelling tasks in class. The Teacher Diagnostic Competence Indicators (TeDCI) was developed to assess teachers' diagnostic competence, based on the framework of difficulties students encounter during the modelling process (Wess et al., 2021). The sample comprised 97 mathematics teachers for 8th and 9th grades who took part in a professional development program for modelling instruction. The program involved practical classroom enactments of mathematical modelling tasks with theoretical learning. Data was collected through self-reports, along with three case studies of teachers, that included semi-structured interviews and observations on the teachers enacting modelling tasks in class. Findings suggest that teachers identified most challenges retrospectively in the *understand simplify* phase (39.8%), and objectively in the *mathematise* phase (27.8%). Prospectively, teachers anticipated challenges in the first two phases solely, highlighting the importance of the diagnostic competence of teachers in these phases. The theoretical significance of this study lies in its contribution to the literature on the diagnostic competence for teachers in the modelling process, particularly in the context of real-time modelling instruction.

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CREATIVITY AS A PATH INTO A GROWTH MINDSET

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Students' broader learning dispositions and their approach to mathematics education, that is, their mindset toward mathematics, bear a significant impact on their academic achievements, self-concept, and professional attitudes (e.g., Dweck, 2006). Dweck (2006) identified two types of mindsets: fixed and growth. Students with a fixed mindset believe that qualities and abilities are innate, often feel the need to prove themselves, may fear failure, and tend to pursue performance-oriented goals. Students with a growth mindset are "learning focused." When encountering a challenge, they are motivated to learn because they believe that qualities and abilities can be developed.

The current study sought to investigate the possibility of promoting a growth mindset by engaging students with creativity promoting tasks, such as open-ended and multiple-solution tasks. Mathematical creativity may be found when students use flexible and versatile thinking to solve mathematical problems in innovative ways (Leikin, & Pitta-Pantazi, 2013). The study involved four 7th grade students who completed pre- and post-tests to evaluate their mindset, and pre- and post-tests to evaluate their mathematical creativity. In between, students participated in three sessions, where they worked as a group, solving open-ended tasks. The researcher's role was to encourage group work. Individual interviews were conducted at the end of the study to further understand the effects of the intervention on students' growth in creativity and their shift towards a growth mindset.

Results indicated that the creativity enhancing intervention led to an increase both in creativity and in students' growth mindset and a decrease in their fixed mindset. One of the students shared in her interview: "I thought it was impossible [to develop creativity] but now I think that if you make an effort, you can develop creativity... You don't have to be born with talent; you can improve over time." This student has gone from feeling it was impossible for her to be creative in mathematics, to believing it is possible by trying. These findings offer valuable insight into the relationship between mathematical creativity and a growth mindset. The study identified several factors that contributed to these results, including the nature of tasks, feedback, and a tolerance towards mistakes. The findings provide a basis for adapting teaching processes to increase both creativity and a growth mindset.

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FIRST- AND SECOND-GRADE PROSPECTIVE TEACHERS RECONSTRUCTING A DEFINITION OF POLYGON DIAGONALS

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The study's purpose is to examine how prospective first- and second-grade mathematics teachers define the concept of polygon diagonals, how they reconstruct their definitions, and how their conceptual images change over time as they align them with the correct definition. Polygon diagonals were chosen due to their recognized complexity and importance. For this study, 23 prospective first- and second-grade teachers participated in a two-part intervention. During two 90-minute meetings, they analysed mathematical events that involved a conflict that could be resolved using a precise mathematical definition of the diagonal. Data was collected through pre-questionnaires and post-questionnaires as well as observations of the class discussion during the two interventions. The researchers used a mixed-method approach and analysed how the participants’ concepts evolved based on Toulmin’s model (2003). The study findings indicated that, prior to the intervention, all participants provided incorrect definitions in the pre-questionnaire (57% insufficient definition, 43% included non-critical attributes) because they relied on their conceptual image rather than their conceptual definition, leading them to include non-critical attributes. When the participants initially engaged with the mathematical events, which included identification examples and non-examples of a diagonal, their claims highlighted their lack of awareness of the gap between the prototype example of the diagonal and the analytical aspect arising from the definition, despite the diagonal definition being provided. However, following the argumentative discourse that we monitored, the participants were able to successfully identify critical attributes and exclude irrelevant attributes. The post-questionnaire showed a significant improvement in the participants' understanding, with 87% able to correctly define whether it was minimal or non-minimal. Based on these findings, which align with prior research (e.g., Haj-Yahya, 2021), it is recommended that future research focus on analysing mathematical events using only definitions as the deciding factor in the identification of examples and non-examples of geometric concepts.

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THE EFFECTIVENESS OF TEACHER PROFESSIONAL LEARNING COMMUNITY OF SELF-REGULATION LEARNING FOR JUNIOR HIGH SCHOOL MATHEMATICS TEACHERS

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The Organization for Economic Co-operation and Development (OECD) proposes the 2030 learning framework, which particularly highlights the student's initiative and the support of systematic collaboration for learning. One of the three core competencies stressed in the general curriculum guidelines of 12-year Basic Education in Taiwan is "learning from autonomous actions". The teacher plays a crucial role in the student's self-regulated learning [SRL]. However, there are very few teacher professional development programmes promoting teacher's self-regulated teaching [SRT]. This study intends to use dialogue analysis as an approach to explore the changes in junior high school mathematics teachers' beliefs about SRT and students' SRL during the process of participating in a teacher professional learning community [PLC].

This study establishes a teacher PLC which implements the "Lesson Study [LS]" in three stages: (1) Plan: teachers collaborate on the design of seventh-grade mathematics SRL lessons, discuss the SRT strategies and the evaluation of teaching effectiveness. (2) Teach and Observe: teachers implement the SRL lessons, and apply the LS observation tools in the classroom observations. (3) Reflection: teachers discuss their observation records for improving the teaching effectiveness, and provide feedback on the design and content of SRL teaching materials. Data collection includes classroom observation records, verbatim transcripts of community activities and teacher interviews, and the pre-and post-tests of the Mathematics Autonomous Learning Scale. This study uses the analytical framework of mathematics classroom teaching research developed by Chen, et al. (2021) to explore teachers' changes made by the PLC, and applies the dependent sample t-test to compare the pre-and post-test results. The results reveal that the teacher PLC may foster participants' knowledge of SRT, transforming their beliefs from teacher-centred to student-centred, and didactic teaching to posing questions and providing feedback to the student. The pre-and post-test analysis indicates that students' SRL abilities get benefits from teacher's SRT.

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MONOLOGICAL AND DIALOGICAL ASPECTS IN THE WORK OF MATHEMATICIANS

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Mathematics as a discipline has a monological image, and is perceived as an objective, absolute, infallible body of knowledge, in which the logical properties alone suffice to establish impeccable mathematical knowledge, and no reference to human agency or social domain is needed (Ernest, 1994). At times when the notion of dialogue has become commonly used in many disciplines, the current research aim is to ask whether the monological image of mathematics does in fact describe the domain. Perhaps mathematical knowledge has dialogic features as well, such as that mathematical knowledge arises in a human dialogue, in a chain of questions and answers in which there is no one correct predetermined answer or meaning (Wegerif et al., 2019).

A qualitative study was designed, and semi structured interviews were held with five faculty members of mathematics departments in several universities. The interviews dealt with describing how mathematicians perceive the domain and their everyday work practices. It focused on the ways they conduct mathematical research; work and communicate with colleagues; and guide research students. The data was analysed with the aim of identifying monological and dialogical aspects in the mathematicians' views of mathematics and their working practices.

In this communication we examine some of the mathematicians' utterances and describe how they perceive the subject domain and act within it, from monological and dialogical points of view. We found that the structure of the subject domain as it is portrayed in the interviews point at a monological core, with inference rules which are perceived as fundamental truth, and theorems written in the language of formal logic. But the core is wrapped by a dialogical process that yields those proofs, and is carried out through a rich dialogue with the mathematical community. Mathematicians choose to present the dialogical process in monological terms in order to grant the theorems a truth status, and to meet their own expectations of the domain.

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PRE-SERVICE MATHEMATICS TEACHERS' SPECIALISED KNOWLEDGE FOR TECHING THE CONCEPT OF A FUNCTION

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The concept of a function is a very important concept in both school and university mathematics. A comprehensive synthesis of literature on the topic suggests that knowledge of functions forms a strong foundation for studying mathematics at high-school and undergraduate levels. While there is a wealth of research in the literature on this topic, not much has been done to measure pre-service teachers' (PSTs) specialised content knowledge (SCK) of functions (Ball et al., 2008). We argue that deep understanding of the function concept equips PSTs with subject matter knowledge that would manifest in their teaching and thereby enable them to properly identify and rectify student difficulties in the classroom.

The aim of this study was to examine PSTs' SCK of functions. Considering the complexity of teacher knowledge and recognising that it cannot be fully measured using a single instrument, we used a multiplicity of research tools in a sequential explanatory mixed methods design. The design included a paper-and-pencil text, semi-structured interviews, vignettes, an activity of creating lesson plans and classroom observations. To adequately examine SCK of PSTs, this short oral communication reports results from a written test that covered concepts including definitions, inverse, composite and different representations of functions.

The sample comprised 150 third- and fourth-year PSTs from two Zambian public universities. Results from the test shows that majority PSTs that participated in this study exhibited low proficiency in their SCK of functions. PSTs demonstrated sufficient knowledge of symbols, domain and range, and were able to perform calculations involving inverse functions. However, they were not proficient with definitions of inverse and composite functions. In addition, PSTs manifested challenges when dealing with problems that required them to move between different representations of functions and exemplification of composite functions using real life situations. They could hardly interpret graphs of functions. This result can have implications for classroom practice because teacher knowledge and level of proficiency impacts one's ability to effectively teach the concept of a function. Clearly, more research focused on PSTs' SCK of functions is needed.

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EYE-TRACKING VISUAL AND TEXTUAL INFORMATION – WHAT MATTERS IN BAYESIAN SITUATIONS?

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INTRODUCTION & THEORY

People's statistical thinking can be analyzed by multiple methods such as questionnaires or tests, but also eye-tracking or think aloud techniques are promising valuable additional insights into people's thinking processes. For instance, Bruckmaier et al. (2019) examined participants' eye movements in Bayesian Reasoning situations based on 2x2 tables filled with natural frequencies, but without providing additional textual information. However, multimedia theory suggests that simultaneously providing text *and* visualization would lead to better performance (multimedia-effect; Mayer, 2021). Furthermore, previous research indicates that when presenting both information formats longer fixation times are expected for the visualizations than for the text (Malone et al., 2020). This leads to the research question which kind of information (and why) participants prefer when statistical information for Bayesian Situations is simultaneously presented textually *and* by visualization (2x2 table). Our corresponding hypotheses are as follows: (H1) The given visualization (2x2 table) is used more intensely in terms of time for information retrieval than the text. (H2) The visualization is preferred to the text because of the already pre-structured data.

METHOD & RESULTS

Participants are given various Bayesian tasks where the information is presented simultaneously as text and visualization (2x2 table). The eye movements are recorded by an eye-tracker as well as participants thinking processes by sound recording (think aloud). The study will be conducted in May 2023 and the results are presented at the PME conference.

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MODELS AND METAPHORS: CHANGING PERCEPTIONS OF MATHEMATICS IN PRE-SERVICE TEACHERS WITH MATHS ANXIETY

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Maths anxiety (MA) is a persistent fear, tension and/or apprehension related to situations requiring maths (Ramirez et al., 2018). MA experienced by teachers can have consequences for student outcomes, since a negative correlation between teacher MA and pupil attainment has been demonstrated (Ramirez et al., 2018). Opportunities to reflect upon personal experiences and emotional responses may reduce the effect of MA on trainee teachers and the subsequent impact on pupil achievement (Foley et al., 2017). Strategies for reducing the effects of MA on pre-service primary teachers were the focus for this action research project, with the following research questions:

- How can pre-service primary teachers with MA be supported to reflect upon their own perceptions of their attitudes towards mathematics?
- Which strategies might help to reduce MA in pre-service teachers and increase their self-efficacy and confidence in doing mathematics?

This paper focuses on 24 trainee teachers from two different PGCE cohorts, who expressed a lack of confidence in either working with, or teaching, mathematics. At the start and end of their 9-month course, participants were asked to construct models of mathematics using Lego Serious Play and to complete questionnaires focused upon their self-perceptions of, and feelings towards, mathematics and mathematics teaching. A range of coaching strategies were integrated into the participants' normal maths pedagogy sessions, alongside additional tutorials. Participants were offered additional coaching support for teaching mathematics during three 8-week school placements.

Participants' models were analysed using conceptual metaphor theory. Pre-intervention, these models revealed a disconnected view of mathematics, with barriers and obscured views of mathematical connections. Post-intervention, a more centrally situated, connected view of mathematics was revealed, which supported greater self-efficacy. Participants reported increased confidence in their own mathematical understanding and in their teaching of mathematics.

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IN-SERVICE MATHEMATICS TEACHERS' CONCEPTIONS OF REASONING AND PROOF

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The contribution identifies similarities and differences in conceptions of reasoning and proving (R&P) in school mathematics by Slovak and Czech teachers at lower secondary schools. In both countries, curricula aim to increase pupils' argumentation competencies and critical reasoning skills. One of the factors influencing the inclusion of R&P into mathematics lessons is the teacher's understanding of R&P and its role in mathematics and its education (Knuth, 2002). A questionnaire was prepared with other researchers involved in the MaTeK project (projectmatek.eu). Follow-up semi-structured interviews with in-service teachers focusing on using resources when planning and enacting mathematics lessons were conducted. Both parts of the study comprise questions concerning teachers' use of resources in general and when teaching R&P, conceptions of R&P, and some demographic questions (Cakiroglu et al., 2023). This contribution pays attention to the answers concerning in-service teachers' conceptions of R&P.

MaxQDA was used for open coding and interview analysis. Identified similarities among answers from both countries are (a) in the way R&P is practiced in a classroom (e.g., as problem-solving, description or explanation of procedures, productive talk, seeing the logic behind the steps), (b) beliefs that not all topics are suitable for R&P, (c) that proof is not suitable for some pupils (for different reasons). The differences observed were in socio-mathematical norms. In Slovak answers the teacher explains, justifies, and comments the solutions when R&P task is present (teacher is active). In contrast, Czech respondents consider classroom discussions and knowledge development in groups more beneficial when doing R&P (pupils are active).

ACKNOWLEDGEMENT

This article was written as part of the H2020 project Enhancement of research excellence in mathematics teacher knowledge, acronym MaTeK, no. 951822.

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THE EFFECTIVENESS OF LEARNING PROGRESS MONITORING: A META-ANALYSIS

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Individualized, adaptive, evidence-based, and learner-centered teaching requires continuous monitoring of students' learning status and individual learning progress. A promising approach that allows monitoring and analyzing an individual's learning over time is *learning progress monitoring* (LPM) (Deno, 2003; Fuchs, 2004). This involves repeated, short tests on the same content. Individual learning progress can be measured and reported to teachers (and students), who can adapt their teaching (or learning) based on this formative feedback. Up to now, there are only a few studies on the effectiveness of LPM in mathematics and beyond, which have produced a heterogeneous and rather unsystematic body of research due to different settings and implementations of LPM (Foegen et al., 2007).

Although initial findings suggest that LPM may positively impact student achievement, there is also a wide heterogeneity in prior results. Therefore, we are currently conducting a meta-analysis to answer the following questions: (i) To what extent does LPM impact students' development of scholastic skills in mathematics and beyond? (ii) To what extent does the effectiveness of LPM depend on characteristics of the sample, characteristics of the LPM, and additional instructional measures?

To answer both questions, we conducted a systematic literature search, which yielded 23 studies with 6,677 students. Preliminary analyses based on a random-effects meta-analysis show that LPM effectively supports students' skills in mathematics and beyond, resulting in a medium positive effect of LPM. Results further underline that the effectiveness of LPM varies based on characteristics of the sample, characteristics of the LPM, and additional instructional measures, so these have to be carefully considered in the planning and analysis of LPM. Results are discussed, and implications for future research are highlighted.

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HOME NUMERACY ACTIVITIES PERFORMED BY DEAF AND HEARING CHILDREN: AN EXPLORATORY STUDY

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Home numeracy activities play an important role in the development of mathematical skills in hearing children (Blevins-Knabe, 2016; Marinova, Reynvoet & Sasanguie, 2021). However, little is known about these activities when performed by deaf children. Thus, this investigation sought to analyze the differences and similarities between mathematical activities performed by deaf and hearing children at home. Due to the COVID-19 pandemic, the data were obtained through observations made by the parents themselves and reported through remote interviews. The participants were Brazilian elementary school children (10 oral deaf and 10 hearing children). The activities were analyzed according to the frequency with which they occurred, and the mathematical knowledge required to perform them. These were of different types: recreational, culinary, school tasks, use of money and conversations about numerical situations. Although both groups of participants experienced the same types of activities, recreational and school activities were the most frequent among the deaf children, while conversations were the most frequent among the hearing children. The deaf children performed fewer activities and interacted with a smaller number of people than the hearing children. Their mathematical experiences at home seem to be less frequent and less varied than those of hearing children. Further research is needed to examine the relations between home numeracy and deaf children´s number shills.

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READING AND CALCULATING IN WORD PROBLEM SOLVING

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Word problem solving combines linguistic and mathematical processes, including reading and calculating. However, it is still unclear how strongly these processes overlap, e.g., if calculations occur parallel or serially during reading. Also, it is unclear how much the mental representation created during reading and the mental representation in which calculations are integrated (e.g., Thevenot et al., 2007). In the present study, we present data from an experiment that investigated the overlap between reading and calculating based on the observation of eye movements during word problem solving.

We systematically varied three-line word problems by manipulating a) the lexical difficulty of two words which were irrelevant for calculations and b) the numerical difficulty of the word problem which was irrelevant for reading. We analyzed the mean fixation duration, which can be interpreted as an indicator for the cognitive effort during processing of visual information (Strohmaier et al., 2020). If processes of reading and calculating occur serially and in delimited mental representations, the manipulation should only affect the mean fixation duration of the varied words and numbers.

Analyses of the pilot data from $N = 17$ undergraduate students show that numerical difficulty influences the mean fixation duration on numbers, but also on the remaining text. Effects of lexical difficulty showed similar trends, but were not significant.

Based on these results and additional data, we argue that eye movements indicate that cognitive processes of reading and calculating overlap during word problem solving and occur in a shared mental representation.

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AN ANALYSIS OF THE SOUTH KOREAN MATHEMATICS CURRICULUM: THE DEMOCRATIC PERSPECTIVE

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South Korea has achieved industrialization in a relatively short period of time, and education plays a pivotal role in the economic development. The democratic dimension of education is vital to transform the way we see mathematics as a subject. If citizens are to participate in a democratic society, they should develop the competence to recognise and understand the use of mathematics by authorities, and the effects of such use. The purpose of this study is to identify the *democratic competence* (Skovsmose, 1990) critical for citizenry development towards a democratic society.

The national curriculum through which the Korean government prescribes content and teaching methods is suggested to be the blueprint for mathematics. The new 2022 national curriculum aims to provide “inclusive and creative education that encourages student agency”. Content analysis was conducted based on 2015 and 2022 mathematics curricula. These documents were then compared to identify from the democratic perspective, any changes that are evident.

The word *democratic* and its forms did not appear in 2015 but appeared 42 times in 2022 curriculum. The objective is clear, “to become *democratic citizens* through [mathematics] learning, problem-solving, and self-examining processes” (Ministry of Education, 2022, p. 6). More elective courses are proposed at the high school level that incorporated sociocultural, economic, and political aspects. As in *Culture & Mathematics*, electoral systems will be studied to explore/ analyze the mathematics behind relevant processes, and to make rational decisions. These courses provide students with opportunities to manage data, to process/ interpret information in social media and their immediate environment, to discuss public affairs and social problems such as fair trade, pollution, and human rights. This study highlights the development of *democratic competence*, a mathematical competence which students need to thrive. Future research will explore how to design teaching-and-learning processes, and to nurture such competence in students in the world of interdependency.

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INFLUENCE OF LINEAR FUNCTION EXPRESSION AND VARIABLE SYMBOLS ON STUDENTS' PROBLEM SOLVING

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The linear function is one of the important learning goals for secondary school mathematics (NCTM, 2000). Students have to understand the meaning of functions and the transformation between equations and graphs. The study focuses on factors of linear function expressions (e.g., y=5x vs. x=5y) and variable symbols (y=5x vs. s=5h), attempting to explore how the factors influence students' problem solving. We designed a 160-trial instrument in which students were required to first read a function graph and then evaluate if the graph was equal to a given function equation. The subjects for the study were high school students who were grouped into the excellence of mathematics (EM) and non-excellence of mathematics (NEM). An ERP methodology was adopted to analyse students' brain activities.

Behavior data analysis showed an interaction between the factors of linear function expressions and the excellence of mathematics. Analysis of the simple main effect revealed that EM students performed more accurately on standard linear function expression (y=5x) than non-standard linear format (x=5y). EM students spent similar time answering standard and non-standard linear equation problems. No differences in response accuracy and reaction time for correct responses were found for NEM students. For the factor of variable symbols, EM students significantly spent longer time answering the problems with non-familiar variable symbols than NEM students. No difference in response accuracy was found between EM and NEM students. Regarding ERP data, analyses of linear function expressions showed the differences between EM and NEM students. For P1 component, the analysis showed that EM students paid higher attention to problems than NEM students. For P3 component, EM students had a higher cognitive load on answering standard and non-standard problems when compared to NEM students, especially in the frontal and parietal areas of the brains. Analysis of the factor of variable symbols showed similar results. For P1 component, EM students paid more attention to the problems than NEM students. For P3 component, EM students had a higher cognitive load in the central area of the brain when compared to NEM students. Although the behavioral data did not show many differences between EM and NEM students, the ERP analyses did reveal significant differences in their cognitive efforts when solving linear function problems.

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SCIENCE EDUCATION STUDENTS' USE OF UNNECESSARY BRACKETS: EXPLORING THE SPAN OF USES AND REFLECTIONS ON STUDENTS' STRUCTURE SENSE

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In many mathematical areas brackets constitute an essential element of notation and can be considered a structuring element in algebra as well as arithmetic (Linchevski, 1995). Students' comprehension of the structural role of brackets has been studied by researchers using additional unnecessary brackets to emphasise the structure (e.g., Hoch and Dreyfus, 2004). However, students' own introduction and use of unnecessary brackets have not been studied in depth.

We report part of our broader study which explores both the students' use and their non-use of brackets (Güler et al., 2022). In this pilot study, we are interested in exploring the range of the different uses of unnecessary brackets which are introduced by the students and explore how this use reflects students' structure sense. The participants, 40 first-year Science Education undergraduate students, were asked to respond to five items (functions, trigonometry, logarithms, derivatives, and integrals) within approximately 40 minutes. These written responses were then analysed using an inductive thematic analysis approach.

The findings illustrate the students use frequently unnecessary brackets as 31, of the 40 participants, used at least once unnecessary brackets, which shows how widespread the introduction and use of unnecessary brackets is among the students. The findings also illustrate that students' use of unnecessary brackets can be categorised in three different ways that mirror aspects of the students' structure sense: keeping the structure of the initial expression, grouping terms of expressions, and aligning with the rules.

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CONCEPTUAL KNOWLEDGE OF PERMUTATION

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Combinatorics deals with the question of how many different possibilities there are for arranging certain objects. The literature and many schoolbooks classify combinatorial questions according to six different situations (permutation, combination and variation each with and without repetition) for providing students clear calculation procedures. Nevertheless, various studies show that students have problems solving combinatorial issues (e.g. Annin & Lai, 2010). The reason for these problems seems to be a lack of conceptual knowledge (e.g. Lookwood, 2014). But combinatorics is an important part of the mathematics curriculum. Therefore, it is necessary to examine students' conceptual knowledge about the different combinatoric situations. However, since basic concepts about combinatorics have not really been formulated so far, the aim of my just starting research is this formulation in a theoretically and empirically founded way. To do so, I initially gave 55 students (20 at school and 35 at university) the task to formulate among other things a word problem for the following term: $4! = 24$. Based on the answers, I inductively formed categories to classify the given solutions. Whereas a total of 29% formulate a suitable word problem, 50% of the participants did not process a factual situation. The remaining 21% told an inadequate story (10% of the pupils and 34.3% of the students). Those can be classified by the following categories:

Table 1: categories of student responses

category	explanation
C1	word problems for multiplication without using combinatorics
C2	distinction between permutation and combination irrelevant
C3	word problems for addition
C4	exclamation mark of the factorial is defined as a variable

While answers assigned to C1 and C2 are mathematically viable if the combinatorial context is neglected, solutions from C3 and C4 indicate that there is no conceptual understanding of the faculty present. This categorisation is the starting point to specify the other basic situations of combinatorics.

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INTRODUCING PATTERNS TO YOUNG CHILDREN WITH AUTISM

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Although there has been considerable focus on the role of patterns in young children's mathematical development (e.g. Woolcott et al., 2015), there has been little research on how to engage young children with autism in mathematical patterns. This is despite Baron-Cohen's (2020) theory that children with autism are hyper-systemisers, who are particularly likely to engage with patterns and patterning.

This study is a form of practitioner inquiry (Cochran-Smith & Donnell, 2006), in that an experienced teacher and her Deputy Head collaborated with a university researcher to examine the teacher's practice, as they investigated the research question: In what ways did the teacher utilise her knowledge of patterns to engage and extend the mathematical thinking of her young children with autism? The data in this study emerged from a thematic analysis of three semi-structured online interviews. All three participants in the interviews had been involved in a project with early years teachers that developed our understanding of developmental trajectories of patterns and useful activities to engage young children in patterning.

Four themes emerged from the data, patterns in: routines; familiar stories and songs; play; and experts. The first two themes involve common practice amongst early years teachers, but it was only with the knowledge gained from the pattern project that the teacher was able to interpret these activities as important precursors to mathematical learning. The last two themes concern how the teacher enhanced her children's mathematical learning through her awareness of patterns and the developmental trajectory of pattern learning. This study shows how enhancing the pattern knowledge of teachers can impact the mathematical learning of children with autism.

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GROUNDING THE PROCEPT OF POLYNOMIALS OPERATION THROUGH EMBODIED INTERACTION

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Gestures can be an effective mediator for mathematics teachers to explain abstract mathematical concepts or demonstrate complex procedures because they can be provided with many cognitive functions, such as guiding students' attention or generating metaphors with conceptual mapping (Robutti, 2020). Furthermore, some cognitive psychologists claim that abstract mathematical thinking and concepts result from the interaction among the human brain, body, and external environment based on the theory of embodied cognition (Castro-Alonso et al., 2015). Therefore, this study compares the learning effects of embodied interaction, dynamic visualization, and static illustration for seventh-grade students when they learn how to multiply polynomials. A quasi-experimental design was adopted in this study. One hundred twenty-two seventh-grade students were randomly assigned to groups demonstrating embodied interaction, dynamic visualization, or statics illustration. The main results show that: (1) All students' correct rates in the posttest show significantly better than those in the pretest, no matter what students learn with embodied interaction, dynamic visualization, or static illustration. (2) The learning improvement of students with embodied interaction is significantly better than those of students with dynamic visualization or static illustration. (3) There is no significant result on the perceived learning difficulty between students with embodied interaction and students with dynamic visualization or static illustration. (4) There are no significant results on the mental efforts, self-efficacy, learning strategies, and degree of the initiative between students with embodied interaction and students with dynamic visualization or static illustration. Finally, suggestions and applications for embodied interaction in mathematical teaching and learning are proposed based on our results and reflections.

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LEARNING GEOMETRY REASONING WITH EMBODIED DYNAMIC VISUALIZATION

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Geometric activities usually involve three cognitive processes: visualization, construction, and reasoning (Duval, 1998). In particular, learning geometry reasoning through vision can reduce cognitive load and build the perception for formal reasoning. In addition to the static visual demonstration, elaborating on dynamic visualization or even incorporating the viewpoint of embodied cognition (such as gestures) will be helpful to the cognitive process of visualization (Castro-Alonso et al., 2015). This presentation illustrates the effects of geometric reasoning with embodied dynamic visualization. Therefore, this study adopted a quasi-experimental design, and the sample comprises 235 seventh and eighth-grade Taiwanese students. All participants were randomly assigned into three groups with different demonstration forms: static illustration, dynamic visualization, and embodied simulation. Besides, the cognitive diagnostic instrument was developed based on three levels of information organization in geometric reasoning: global, local, and micro (Duval, 1998), to assess students' learning outcomes. Results indicate that: (1) Students' understanding of geometric reasoning was promoted significantly after asking students to study learning materials demonstrated by embodied simulation or dynamic visualization. Moreover, this effect was mainly reflected in the low mathematics achievement students. (2) Demonstrating learning materials with dynamic visualization was more effective in helping students grasp the context and structure of the geometric reasoning process. (3) Students with embodied simulation showed more better on the overall performance and the micro level of organization in geometric reasoning than those with dynamic visualization or static illustration. Further research is needed to investigate the cognitive mechanisms of embodied simulation in geometric reasoning.

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DO FUTURE PRIMARY TEACHERS IN TAIWAN HAVE SUFFICIENT BASIC COMPUTATIONAL CAPABILITIES?

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The IEA Teacher Education and Development Study in Mathematics (TEDS-M) found a wide range of achievement among future primary teachers across countries, with Taiwan at the top of the list (Tatto et al., 2012). However, after a decade, do we still have such results? Additionally, it is worth noting that the TEDS-M survey does not emphasize the basic computational capabilities that are highlighted in primary school mathematics curriculum; rather, it assesses the overall content knowledge of mathematics. Therefore, the purpose of this study is to explore the basic computational capabilities of future primary teachers in Taiwan and compare them with those of non-future primary teachers. The study seeks to determine whether the infrequent use of arithmetic involving fractions and negative numbers in daily life, along with the availability of calculators, reduces the computational capabilities of future primary teachers after they leave school. In addition, the study aims to analyse error rates to identify misconceptions and error patterns that could provide teacher education institutes with ideas for teaching mathematics. This fact underscores the necessity and significance of this study.

This study utilized a questionnaire consisting of fifty items, which assessed four domains: integers, fractions, decimals, and percentages, to collect quantitative data. The study participants were drawn from Taiwanese public universities, including future and non-future primary teachers. Over one hundred questionnaires were collected with both groups consisting of over thirty individuals. The study compared the performance of future primary teachers with a control group of non-future primary teachers. The data are analysed by descriptive statistics.

The results indicate that there are no significant differences in basic computational capabilities between future primary teachers and non-future primary teachers; however, errors were observed in both groups. For instance, around 70% of the subjects did not include units in their answers. Additionally, in a question involving fractions and decimals "Convert $\frac{5}{32}$ to a decimal" around 27% of the subjects provided an incorrect answer such as 0.156 or ≈0.16 instead of the correct answer 0.15625. This could be indicative of the educational environment in Taiwan. During the presentation, several examples will be discussed, and these can be explored in greater detail.

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EVALUATION OF MIDDLE SCHOOL MATHEMATICS TEACHERS' TEACHING EXPERIMENTS IN TERMS OF GRAPH LITERACY

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INTRODUCTION

In our world, which is surrounded by data day, the importance of individuals' ability to understand interpret, draw conclusions and question the statistical information (Gal, 2019), on TV, magazines, newspapers and internet sites is also emerging. In our daily life, statistical information is mostly presented via graphs through written or oral media. In this context, present study, it is aimed to evaluate the teaching of middle school mathematics teachers about graphs in terms of graph literacy.

METHOD

In this study, the case study, which is one of the qualitative research method was used. The study was carried out with 6 mathematics teachers who taught at the 7th grade level. In the study, related to graph literacy practices an observation form consisting of 18 indicators were developed. In-class observations and interviews were the data collection tools of the study. In-class observation data of teachers were coded within the scope of structured observation form and analyzed qualitatively.

CONCLUSIONS

In our research, it was revealed that classroom practices of the teachers mostly focused on practices about reading between data with the context and asking questions. And they referred to for the aspect of making graph drawings by changing the data and interpreting them at least. By the interviews, it was seen that there were shortcomings in the graph literacy skills of the teachers. Within the scope of these results, it is recommended to provide in-service trainings that will contribute to the graph teaching processes of teachers.

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NON-ROUTINE PROBLEM SOLVING AND CREATIVITY AMONG TALENTED MATH STUDENTS FROM A MULTI-AGE PERSPECTIVE

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One of the most significant characteristics discussed in the literature in the context of talented students is creativity. Leading researchers in the area of problem solving (PS) and creativity have argued that non-routine PS, and in particular Multiple Solution Tasks (MSTs), can be an assessment of individual creativity (Leikin, 2013). This study deals with PS amongst 420 talented students aged 10-18 who take part in the prestigious mathematical enrichment program known as "Kidumatica". Our goal was to examine the students' solutions from a multi-age perspective. Data was gathered from students' products and teacher observations during a series of workshops devoted to 10 non-routine problems with multiple solution paths. Quantitative and qualitative analysis were based on previous models in the area of creativity and PS, specifically in MSTs and provides a full assessment per student for each problem. The motivation for this study is based on previous research, in which we found solutions that were common amongst students aged 9-10, that had not been discovered among similar students who were older (Uziel & Amit, 2016).

Analysis of our findings reveals a troubling phenomenon: As the age of students rises, they are less prone to looking for creative and holistic solutions when solving problems, and more likely to be “held hostage” by their habitual use of algebra. This study reflects an issue that we should take into account: Adult students suffer from ‘thinking fixation’ which eliminates flexible thinking and prevents creative approaches to PS. The NCTM standards (2000) have recommended that students develop their ability to connect different mathematical ideas by solving a single problem in several ways. This study confirms the need to implement the NCTM' recommendation in all classrooms and especially among secondary schools students.

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DIFFERENCES IN STUDENTS WHO EXCEL IN SCHOOL MATHEMATICS IN TAIWAN AND ISRAEL

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Both the Israeli and Taiwanese educational systems foster excellence in mathematics (EM) on school. However, the two countries differ in their instructional approached when teaching mathematics. In particular, there are different streaming practices that lead students to study in high level mathematics in high school. In Taiwan, students pass entrance exams for elitist schools offering university admission. Taiwanese education is influenced by Confucianism, emphasizing practice, effort, and extensive examinations, leading students to attend "cram" schools for mathematical training. (Leung, 2002). In contrast, in Israel, European education system is dominant and the level of teaching is prescribed by the matriculation exam.

To examine the difference between EM students in Israel and in Taiwan we performed comparisons between EM and NEM (non-excelling in mathematics) students between the countries. We examined levels of mathematical competencies using SAT-M (Scholastic Assessment Test in Mathematics) and general giftedness using RPMT (Raven’s Advanced Progressive Matrices Test).

We found that Taiwanese EM students had higher SAT-M and RPMT scores as compared to Israeli EM students (SAT-M [$F(1,156) = 241.656$, $p < .001$] and RPMT [$F(1,156) = 27.554$, $p < .001$]). At the same time, in NEM groups, there was no difference between SAT-M scores in Israel and Taiwan, while the RPMT scores of Israeli NEM students were higher than those of NEM students in Taiwan.

Studies about the connection between general cognitive ability and mathematical competence reported that SAT-M scores significantly correlated with RPMT scores and SAT-M scores were predicted by RPMT scores (Rohde & Thompson, 2007). In our study, we also found that RPMT scores significantly predicted SAT-M scores with larger variance for the Taiwanese sample. This finding suggests that general cognitive ability makes an important contribution to mathematical achievement. Therefore, it is not surprising that Taiwanese EM students had high RPMT scores in addition to high SAT-M scores.

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THE INITIAL STAGE TO CONSTRUCT THE CONCEPTS OF TIME: OBSERVING A CLOCK FROM THE VIEWPOINT OF PERSONIFICATION

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The task of "Time and Duration" is one of the contents of mathematics that children immediately after entering school find difficulty to interpret. The concepts of time include point of time (“measured” or labelled by clocks), and durations (measuring elapsed time), It is very abstract ideas about the nature of time as a flowing direction (Harries, 2008). However, we tend to be unaware of the mathematical difficulties behind it since we-from kindergarten children to adults-are too involved in time daily. One mathematical difficulty in developing time concepts is that children who have not yet learned the decimal system well have to tackle tasks that involve sexagesimal system. Thus, it is important that the stages for mathematically constructing the concept of time are revealed. This study especially clarifies the initial stage. The research questions are:1) How do kindergarten children perceive “Time and Duration”? 2)Are there new indispensable mathematical activities for constructing the concept of time in kindergarten and elementary schools?

The sample comprises 22 five old children of the affiliated kindergarten of the public university who have better mathematically ideas and explanations than them of the same generation. The teacher’s intervention took place about ten minutes in “the scene for deciding of a time to start and stop outdoor gaming -the gaming time was from nine thirty-five to ten twenty-” by using some analog clocks. Data for analysis was collected with a video camera and analyzed in the qualitive approach from the viewpoint of *personification*. The tendency of children to engage in personification could potentially be observed in abstract entities, such as numbers (Matsuda, et.al, 2018).

In data analysis we found two points out. One was that the children’s numerical expressions were mostly applied to the "position of the long hand" on the analog clock, on the contrary, the "shorthand" was rarely discussed. Another was that the children personified the two hands of the clock, observed their movements, and discussed characteristics of the movement using the personifies. These suggested that it is necessary to observe the writing on the clockface and the movement of the hands of the clock, before reading a point of time or learning about time through a number line.

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POTENTIAL OPPORTUNITIES AND CHALLENGES IN THE INTEGRATION OF EXECUTIVE FUNCTION PROCESSES IN MATHEMATICS EDUCATION

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Executive functions (EFs) are used regularly in productive mathematical activity (cite). Awareness of approaches for the integration of EFs in mathematics education can inform the design, adaptation and use of instructional resources and professional learning activities. Bull and Lee (2014) argue that EFs assist students in problem solving through suppression of irrelevant strategies and shifting between solution strategies. Recent attention given to Universal Design for Learning has motivated interest among math educators regarding how EFs might be leveraged to support problem solving, strategy development and self-regulation (CAST, 2018).

Three core EFs identified by Diamond and Lee (2011) – i.e., cognitive flexibility, inhibition and working memory – were described and exemplified in lesson sequences and games addressing middle grades mathematics instructional goals co-designed by our project team and participating teachers. In this study we examined teacher enactment of and student engagement in observed EF integrated lessons piloted by 10 teachers in four middle schools in the United States. Observations of teachers' and students' business as usual (BAU) and EF integrated (EFI) lessons were compared using the MCOP2 protocol and field notes.

RESULTS

Observations of teachers' BAU mathematics lessons for introductory statistics lessons were characterized by greater emphasis on mathematical structure, precision of mathematical language, and teacher talk focused on "lower order" knowledge-based questions and responses. In contrast, teachers' EFI lessons were characterized by higher levels of student engagement as they reasoned about statistics concepts and data in relation to familiar problem contexts. A wider range of student strategies were shared by students and focused on explanation of strategies rather than just sharing the answer. MCOP2 data and findings will be shared in greater detail in the presentation.

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PRESERVICE MATHEMATICS TEACHERS NOTICING IN CHINA

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Teacher noticing is a crucial component of a teacher's professional competence. Researchers proposed that teacher noticing shall consist of perceiving typical events in an instructional setting, interpreting these activities, and responding to students' thinking (Amador et al., 2021; Kaiser et al., 2015). However, most of the current research and developed frameworks have been conducted in western educational contexts, there is little evidence discussing teacher noticing in non-western contexts. The purpose of this study was to examine pre-service mathematics teachers' (PSTs) noticing in China. The sample consisted of 127 PSTs with a major in mathematics teaching. Participants were invited to watch a primary exemplary mathematics lesson. Data collection includes PSTs' perceptions of the lesson, the classroom events they noticed, how they interpreted the events, and how they responded to the events. The framework proposed by van Es (2011) was used to differentiate the noticing skills.

The findings indicated that both before and during the lesson observation, PSTs showed more concerns for two subjects, which were students (74.36%) and teachers (67.89%), but seldom mentioned any other subjects involved in the classroom. Most PSTs were able to describe and explain the classroom events (35.24%). Some PSTs were able to describe, evaluate, and explain (32.38%). It showed that almost half of the PSTs showed a mixed level of noticing skills to highlight meaningful events or details in the classroom with a reasonable explanation (48.57%). However, PSTs experienced difficulty in providing their own response methods. More than half of the PSTs could only manage to describe their response methods (56.25%). Most PSTs could only reach the baseline level (90.63%), and their reporting usually lack concrete evidence of content. In summary, perspectives of PSTs' noticing are still relatively limited and cannot focus on every aspect of the classroom. They have mastered certain noticing skills, but they cannot fully demonstrate them in the process of practice. How to guide PSTs to use their noticing skills needs further study.

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ASSESSMENT OF REFLECTION SKILLS EMBEDDED IN MATHEMATICAL LITERACY

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This study aimed to propose a conceptual framework for assessing reflection skills embedded in mathematical literacy (REM) and develop an instrument to assess it. The first dimension is based on the phase of modelling process in PISA 2021 mathematical framework, including the three components of formulation, employment and interpretation/evaluation. Based on Dewey's (1933) notion of reflective thinking and the contrast of it with critical thinking, the second dimension of REM includes three subskills, perception of rationality, explanation of rationality and judgement of optimality. The two dimensions of REM are shown in Table 1.

Problem Solving Reflection skill	Formulate	Employ	Interpret/Evaluate
Perception of rationality	To determine whether a situational information, mathematical process or situational result is reasonable.		
Explanation of rationality	To provide reasons for their own perception of a situational information, mathematical process or situational result.		
Judgment of optimality	To judge which situational information, mathematical process or situational result is optimal.		

Table 1: A Framework for Assessing Reflection Skills Embedded in Mathematical Literacy

Forty-five questions of REM in five contexts have been designed and reorganized as four forms. Each form was completed by around 200 secondary students. The questions in each form were validated by confirmatory factor analysis as a three-factor model. The results of IRT analysis not only justified the overall quality of the questions, but also provided information about the differences in the difficulties of the three subskills. Students in the top level of REM are able to judge which solution is optimal using knowledge-based or evidence-based reasoning. Further investigation is to compare students' REM with their critical thinking skills embedded in mathematical literacy.

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COMPARISON OF EQUIVALENT FRACTIONS WITH DIFFERENT REPRESENTATIONS

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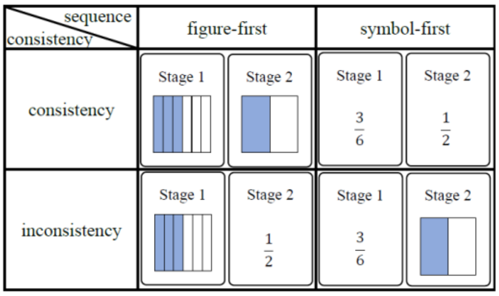

Understanding the representation of equivalent fractions is an important part of mathematics education (Bouck et al., 2017). This study aimed to examine the representation of two factors (sequence and consistency) on the equivalent fraction based on behavioral and brainwave data. The study used an equivalent fraction verification task that asked participants to determine whether two consecutive fractions were equal. The sequence was further divided into two categories, symbol-first and figure-first, with symbol-first indicating that the first representation used as symbol, and figure-first indicating that the first representation used was figure. Consistency referred to the use of the same representation for two consecutive fractions, while inconsistency referred to the use of different representations. E-prime software was used to execute the task, and participants' response accuracy (Acc) and reaction time to correct response (RTc) were used to determine the cognitive complexity of the task. Event-related potential (ERP) techniques were used to collect brainwave activity during problem solving.

The subjects of this study were college students. Behavioral results showed that regardless of whether the representations were congruent or incongruent, symbol-first representations were more accurate and had shorter reaction times than number-first representations, indicating that symbol-first cognitive complexity was lower. Notably, the lowest accuracy and the longest reaction time in the consistent and picture-first condition indicate the highest cognitive complexity. Brainwave analysis of N270 and P300 components. The N270 component refers to conflict processing in error detection, while the P3 component reflects cognitive effort in synthesizing stimuli and reasoning results. Regarding the N270 component, the figure-first had a larger N270 brainwave response, showing that the figure-first had a more extensive error detection cognitive process. Furthermore, the P300 for consistency and inconsistency differed regardless of whether symbol-first or figure-first. The results showed that the cognitive effort of representational consistency and inconsistency is different.

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EXPLORING GRADE 11 STUDENTS' METACOGNITIVE PROCESSES IN SOLVING NON-ROUTINE PROBLEMS IN GEOMETRY

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Metacognition is the process of thinking about thinking. When studying mathematics, metacognition can play an important role in knowledge acquisition, retention and application. The metacognitive behaviours of students during non-routine problem solving are well documented in the literature (Ellestad & Matusovich 2021), however, few of them have compared the metacognitive dispositions of students from different ability groups. This study explores this gap and shed light on the variation in metacognitive behaviours of students of different abilities (low, average, high). A convenience sample of 6 Grade 11 students (2 from each ability group) were selected from a private school. Qualitative data were collected through task-based interviews when students, of the same ability, were solving, in pair, a non-routine problem on geometry. Three interviews were conducted separately for the different ability groups (low, average, high). The interviews were video recorded and transcribed verbatim. The transcriptions were coded and analysed based on Kuzle's (2013) metacognitive framework. It was found that metacognitive dispositions varied with the ability of the students. That is, when working in pairs, high ability students displayed relatively more metacognitive dispositions and low ability students displayed relatively less. Despite showing more metacognitive dispositions than the low ability students, the average ability students were not successful, in solving the non-routine problem, unlike the low achieving students. These observations suggest that high use of metacognitive processes do not necessarily imply successful problem-solving. The findings prompt for more research to further explore the impact of metacognitive dispositions on problem solving ability of students from different ability group. The implications in the teaching and learning of mathematics at Grade 11 should also be explored.

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ADOPTING THE READING TO LEARN APPROACH IN THE TEACHING OF MATHEMAICAL WORD PROBLEMS

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Mathematical word problems (WPs) are generally considered challenging to students, where such challenges are amplified when students study mathematics in their second language. Despite the increased learner diversity in mathematics classrooms and the fact that language plays an inseparable role in the understanding of WPs, very few mathematics teachers are aware of the potential benefits of adopting language learning strategies such as genre-based pedagogy “Reading to Learn” (Rose & Martin, 2011) or making use of culturally responsive teaching (Gay, 2018) in their lessons.

The research presented here aims to understand the impact of integrating the Reading to Learn approach in teaching WPs in a Hong Kong primary school, where mathematics lessons are conducted in English, and how the integration resembles culturally responsive teaching. Two non-native English-speaking teachers and 60 third grade students participated in the study. The students speak a variety of languages at home, including Chinese (n=27), English (n=11), Nepali (n=17), Urdu (n=4) and other (n=1). The intervention took place in six lessons, each lasting 30 minutes. During the lessons, selected components of the Reading to Learn framework, namely detailed reading, sentence making, joint rewriting and individual rewriting, were used. An integral part of the intervention was to invite students to identify and discuss problems related to mathematics in their daily lives and cultural backgrounds, and write WPs by themselves. Students’ diagnostic test papers and worksheets as well as teachers’ lesson plans and reflections through interviews were collected for analysis.

The results from teacher interviews suggest that the process successfully drew upon students’ cultural funds of knowledge, and provided language support for students to articulate their mathematical thinking, leading to meaningful classroom discourse. The findings from students’ worksheets also indicate that students’ knowledge and understanding of how WPs are formed and solved are deepened. Based on teachers’ narratives and student work, the researcher sees the potential in using language learning strategies in multicultural and multilingual mathematics classrooms. Future research will use quantitative analysis of the diagnostic test papers to provide another perspective for further explanation of the impacts of the intervention.

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INTEGRATING TECHNOLOGICAL TOOLS IN MATHEMATICS EDUCATION IN THE CONTEXT OF HYBRID TEACHING DURING THE COVID-19 PANDEMIC

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Hybrid teaching during the COVID-19 pandemic put teachers in the challenging situation of needing to adjust quickly to the use of digital platforms. At the same time, an opportunity was created for change and innovation (Drijvers et al., 2021). Our study examines teachers' experience looking at the aspects of 'learning management', 'mathematical challenge', and 'sensitivity to students' (the 'Teaching triad' model) (Jaworski, 1992). This article focuses on the integration of technological tools, which was one of the main themes that appeared in interviews. The research question: *In which ways were technological tools integrated into mathematics teaching in the high school in the framework of hybrid teaching during the COVID-19 pandemic, and what were the barriers and the catalysts to use these tools according to teachers?*

Semi-structured interviews were conducted for two hours each with 15 leading teachers who taught in a hybrid environment in high school in 2021. For analysis, the teachers' statements were first categorized according to the elements of the teaching triad. Then an inductive analysis was performed. One of the categories that arose under 'learning management' was integration of technological tools. Four types were identified: (1) Integration of mathematics dynamic softwares, (2) integration of independent learning platforms, (3) using technological tools to vary instruction, (4) integration of digital platforms allowing assessment and tracking of both mathematical understanding and affective aspects. Moreover, three categories were identified for factors that allowed or limited the usage of technological tools: (1) linked to the teacher's characteristics teacher; (2) the technology; and (3) to the instructional context. These findings teach us about ways of using technology that were not common before the coronavirus. In the conference we will address the different uses of technology and the factors that supported and/or limited them, as a foundation for learning about future integration of technological tools in both hybrid teaching and "regular" teaching.

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A STUDY ON THE DETERMINATION INDICATORS FOR MATHEMATICAL LITERACY QUESTION: FROM THE PERSPECTIVES OF THE SCHOOL TEACHERS

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Mathematical literacy is the most popular issue of mathematics education in Taiwan in recent years. School mathematics teachers in Taiwan need to design mathematical literacy question (MLQ) to evaluate their students' mathematical literacy. However, what kind of question is a mathematical literacy evaluative question? Although countries have proposed the definition of mathematical literacy, it is well known that the definition and the meaning of mathematical literacy are different around the world. Such differences also affect the assessment of students' mathematical literacy. The purpose of this study is to explore the characteristics and the indicators of MLQ suitable for the teaching scene in Taiwan and try to set up evaluative standards for MLQ to help teachers design MLQs.

The two research questions we try to solve are as follows:

- From the perspectives of the primary and secondary school teachers, what are the characteristics and indicators of MLQ?
- How do we set up the evaluative standards for MLQ based on the indicators?

Our study employed both grounded theory methodology and questionnaire method to explore the characteristics and the indicators of MLQ. This study collected MLQs from various sources, including those released and declared by Taiwan's education authorities and research institutes. Examples of MLQs include those from the Literacy-oriented Assessment samples from the National Academy for Educational Research, and the released exams of PISA. To collect MLQ data, this study used a system sampling method and included 42 items in the questionnaire. Purposive sampling was employed to select the study participants, who were comprised of in-service primary and secondary school mathematics teachers. A total of 68 questionnaire responses were collected and analysed. The research findings on MLQ indicators reveal two dimensions: student characteristics and question representation. The student characteristics dimension comprises five indicators, including motivational triggers, capability of interpreting graphic materials, application of mathematical concepts, capability of multiple problem-solving, and logical thinking. The question representation dimension also consists of six indicators, namely, relevance to daily life situations, open-endedness, logical coherence, interdisciplinary relevance, innovation, and readability. The MLQ evaluation standards prioritize the application of mathematical concepts, innovation, and logical thinking proficiency. Based on these research results, we aim to develop a checklist to assist teachers in designing effective MLQs in the future.

HOW DO TEACHERS ANTICIPATE STUDENTS' RESPONSES IN PROBLEM-SOLVING LESSONS?

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Anticipating students’ responses to a proposed task, by their teachers, appears as an essential stage in various models of lesson plan. Teachers are supposed to put aside the way they would act if they were requested to solve, put themselves in the students' shoes, and try to guess how different students would approach the task (Stein et al., 2008). Although the practice of anticipating is endorsed in the literature, no empirical study took place with the goals to identify its characteristics, and to reveal those ones that will be especially productive in preparing and conducting problem-solving lessons.

Stimulated by the concept of *thick description* taken from the field of qualitative data analysis, this research aims to characterize *thick anticipation* in order to offer a tool for teacher educators to cultivate teachers' use of anticipating as a teaching practice for problem-solving-based instruction.

Ten high school mathematics teachers participated in the study. They were interviewed based on three selected tasks. The interviewees were asked to describe an imaginary solving process in voices of different students, using the *virtual monologue* tool. Full transcripts of the interviews were thematically analysed.

Five characteristics of the teacher anticipations were found: (1) Specificity: Does the teacher refer to a specific student, to a generic one, or to all students as one unit? (2) Mathematical moves: Does the teacher describe stages in detail or just a solution way in title? (3) Organization: Does the anticipation include organizational and communicative aspects? (4) Solver's voice: Does the anticipation include a first-person narrative of the solver? And (5) Variety: How many anticipatory scenarios of different students' solutions does the teacher propose? Research findings show that the more familiar the teacher is with the task, the richer anticipation he or she is able to produce; and in turn, the richer the anticipation is, the more the teacher is inclined to use the task in a real lesson.

We deem the results of the study of interest to teacher facilitators for designing professional development activities aimed at encouraging mathematics teachers to anticipate their students’ responses when preparing problem-solving lessons.

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POSTERS

PME-46

MATHEMATICS EDUCATION

FOR GLOBAL SUSTAINABILITY

CONSIDERATIONS FOR THE STUDY OF TEACHER'S BELIEFS FROM THEIR ACTIONS

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As beliefs must be inferred by teachers' actions, researchers must take theoretical and methodological decisions about which elements of teachers' actions they are going to focus their attention on, and how are those elements related to teachers' beliefs.

Two notions have been used to state a relation between teachers' actions and teachers' beliefs: *goals* as proposed by Schoenfeld's framework (2008) and *social* and *sociomathematical norms* from Cobb & Yackel's framework (1996).

We state that the complementary use of these two notions provide a finer grain analysis of teachers' beliefs and that they both may be analysed by the Normative Dimension given by the Ontosemiotic Approach (OSA) (Assis et al., 2012).

Teachers' goals relate and can be studied by teachers' configuration proposed by OSA, which constitute the norms that teachers are negotiating with their student to motivate them and to assign, regulate and assess their learning. Furthermore, social and sociomathematical norms relate and can be studied by Metanorms, as proposed by OSA, that is, as norms that regulate norms, or norms that remain unchanged for a certain period.

By studying the norms promoted by teachers, researchers have a starting point to analyze the reasons that led to them. While identifying metanorms allows them to access possible explanations of those norms. Both cases allow the inference of teachers' beliefs from their actions.

In this way, taking beliefs as psychological correlates of the classroom social norms (Cobb & Yackel, 1996) that shape the selection of goals (Schoenfeld, 2008), we state that the use of norms and metanorms, as proposed in the normative dimension given by OSA, is a helpful way of inferring teachers' beliefs from their actions.

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MATHEMATICAL PROBLEM POSING AND PROBLEM SOLVING BY ELEMENTARY SCHOOL CHILDREN: A TEACHING EXPERIMENT

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Some studies on problem posing examine how children formulate problems in different situations, and some studies on the topic seek to develop children's ability to formulate and solve problems (English, 1998; Kwon & Capraro, 2021). The focus of this intervention study is the latter. The participants, 42 third-grade Brazilian children, were given a pre-test and a post-test, each consisting of two tasks: to formulate arithmetical word problems and to solve arithmetical word problems. After the pre-test participants were divided into a control group (CG) and an experimental group (EG). The children in the EG had to complete a sequence of activities conducted by the teacher in the classroom: (i) identify which word problems were complete and which were incomplete; (ii) identify the missing part of the incomplete word problems and complete those; (iii) change some of the features of the word problems; and (iv) formulate problems based on pictures showing everyday situations. The children worked individually and in small groups. The teacher encouraged whole class discussions and, at the end of each session, offered a solution to the activities proposed and asked the students to solve the problems they had formulated. No significant differences were found between the two groups in neither task of the pre-test. However, in the post-test, children in the EG performed significantly better than those in the CG in both tasks. Children in the EG were also significantly more successful in the post-test than in the pre-test in the problem-posing task. This study has educational implications for the teaching of problem posing in elementary school.

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FEATURES THAT PRE- SERVICE ELEMENTARY SCHOOL MATHEMATICS TEACHERS USE WHEN IMPLEMENTING THE PBL METHOD

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The teachers and pre-service teachers have faced a double challenge in recent years: implementing changes in teaching patterns and adopting pedagogies that they had not experienced while studying. Teaching using the Project-based learning (PBL) method is an example of this. The method presents the pre-service teachers with the challenge of dealing with a change in the teacher's role (Palatnik, 2022).

The purpose of this study was to examine the features of the teaching opportunities that pre-service elementary school mathematics teachers tended to utilize when using (PBL) in their lessons.

The theoretical framework is based on using a ‘pedagogical map’ to present pedagogical opportunities that are exploited by mathematics teachers when teaching with new technology (Pierce & Stacey, 2013). The map was adapted to examine which opportunities are exploited when teaching mathematics using the PBL method by focusing on two aspects: pedagogical-educational and social-emotional.

Three main pedagogical characteristics have been examined: knowledge, skills, and habits. Additionally, social, and emotional aspects have been examined include opportunities that support competency, autonomy, and the need for connection, relatedness, and safety.

Study methodology consisted of case studies. Ten cases in which pre-service mathematics teachers utilized PBL pedagogical opportunities in their lessons were examined. Their aspects were mapped and analysed using a map of pedagogical opportunities for each pre-service teacher.

Findings indicate that by using the PBL method, students became familiar with changing the role of the teacher. The balance between acquisition and construction is altered as well as emphasizing the importance of process work. Furthermore, pre-service teachers viewed PBL as a method supports the sense of competent and autonomous. This study discusses implications for educators and teacher educators in the integration of the PBL method into teaching mathematics. In addition, a tool for mapping pedagogical opportunities using PBL is presented.

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SUPPORTING PRE-SERVICE TEACHERS' SKILL OF CRAFTING A PEDAGOGICAL RESPONSE THAT BUILDS ON CHILDRENS' MATHEMATICAL THINKING

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Teaching that attends to and builds on students' productive and diverse resources and treats all students as capable mathematical sense makers has been referred to as responsive teaching. To craft a response to deepening children's mathematical understanding, teachers need to engage in pedagogical reasoning drawing on their Mathematical Knowledge for Teaching (MKT), specifically specialized content knowledge and knowledge of content and students. Pre-service teachers' (PSTs) pedagogical reasoning is mostly studied in the later stage of their teacher preparation, in connection with the field experience. Given that PSTs start developing MKT in the early years of the teacher education program, this results in a separation between the learning of mathematical knowledge and the work of teaching.

This study draws on a perspective of teacher learning as interactive; individuals interact with and learn from each other and learning spreads over time (Russ et al., 2016). Individual PST contributes to shared understanding as they formulate questions, propose alternative ways of explanation, and negotiate meanings and actions. In this study, I will investigate how PSTs' mathematical sense-making in a content course focusing on MKT will contribute to their engagement in crafting mathematically based pedagogical responses that elevate children's mathematical understanding. The research question is:

- How do PSTs develop their capacity to craft evidence-based responses to children's mathematical thinking, in tandem with developing forms of MKT in the content class?

This qualitative case study was conducted in a Pedagogical Content Knowledge for Teaching Elementary Mathematics-I, and the sample comprises of 23 PSTs. In this study, the MKT task structure and discussion scaffolding questions were the key interventions used to enhance PSTs' development of collective pedagogical judgment by utilizing their MKT. The preliminary results indicate a cyclical rather than a linear relationship between the skills of attending, interpreting, and deciding how to respond. In the poster, the further result of detailed analysis will be presented.

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IDENTIFYING TEACHERS' NEEDS IN INCLUSIVE MATHEMATICS CLASSES

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Due to society's changing values, we are currently witnessing a fundamental change in the composition of students learning in regular classes with the inclusion of students with special needs in mainstream classrooms (Gilor & Katz, 2019). Lambert and Tan (2019) presented a review of international research on teaching and learning mathematics from kindergarten to the 12th grade, comparing studies of students identified as having disabilities to studies of students without disabilities. They found few studies concerning the way teachers adapt the curriculum to meet the complex needs of students with disabilities. The purpose of the current study was to identify the needs of mathematics teachers in elementary school mathematics classes that include students with special education needs.

Participants were 106 elementary school mathematics teachers who filled in a questionnaire related to their perceptions of including students with special needs in regular mathematics classes and describing the teachers' needs. It was found that more than 50% of the classes in the sample include three or more students with special education needs, such as students with multiple learning disorders, emotional behavioural disorders, and AD(H)D. Half of the teachers in the sample claimed that providing concrete manipulatives to those students would be helpful for a student's inclusion, yet more than 50% of the teachers testified that their school does not have appropriate manipulatives, like base ten blocks. A third of the teachers testified that their school has a special education teacher who teaches those students mathematics by taking them out of the class and teaching them in a separate group. Over half of the teachers said that children with special education needs receive help in language, but not in mathematics. To help include students with special needs in mathematics classes, teachers suggested reducing the number of students in the classes, developing tasks of different levels of difficulty, and placing an additional teacher or assistant to help with the students included in regular mathematics classrooms. From this research we see the need for professional development for teachers that includes theoretical and applied knowledge on including students with special needs in math.

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SUSTAINABILITY IN MATHEMATICAL CLASSES: TENSIONS OF A TEACHER EMPLOYING AN INTERDISCIPLINARY GAME

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The tensions lived by a teacher, Federico, in the design and implementation of a game-based mathematics lesson are analysed. Learning games do not only entertain, but teach skills, knowledge and attitudes. Pan, Ke and Xu (2022) report that: (i) puzzles are the most used ones, followed by strategy, adventure, role play and action games, (ii) many games involve students' low-order skills (e.g., basic computations) with the exception of simulation ones, stimulating high-order skills such as argumentation or modelling (Pan et al., 2022). The game designed by Federico raises awareness about socio-environmental issues concerning the cultivation of cotton, through the simulation of a real situation. Mathematics plays a rather marginal role and this can be in conflict with the importance of the preparation for the final exams, generating tensions for the teacher. Tensions have been characterised as opposing forces (Berry, 2007). Which tensions emerge in the design and implementation of a game-based interdisciplinary lesson? How does the teacher manage them? Federico represents a single case study and is at the beginning of his career. Two semistructured interviews (one (i) before and one (ii) after the lesson) let the tensions emerge from the phases of (i) lesson plan and (ii) lesson implementation and teacher's reflections after the lesson. A qualitative coding method allows to identify the different tensions, and to interpret whether Federico solved, lived with or avoided each tension. The results shed light on the obstacles that teachers face when teaching an interdisciplinary lesson on sustainability issues, and thus that can prevent teachers enacting such a teaching. The first two tensions pivot around the complexity of the real world and the difficulty of the mathematics embedded in the game, and Federico tries to resolve the first one in favour of the students' requests, while he seems not really changing the second one. The third tension refers to the actual possibilities for class management.

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‘DOING DIFFERENCE’ IN MATHEMATICS EDUCATION

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Following a cultural-sociological approach, differences in learners cannot solely be seen as an information that mathematics teachers have to adapt to. Moreover, complex social constructs and subsequent products of external and ascriptive attributions in social interactions regarding individuals or groups go along with this setting. Thus, teaching mathematics becomes a stage for the reality of (re)enforcing these constructed differences. ‘Doing Difference’ provides an analytical frame to trace these processes of constructing differences and to analyse them as products of constructions within social interactions and practices following a specific logic within the field of emergence (Hirschauer, 2017). Situating the empirical base for research in mathematics teachers’ knowledge and their attribution of ir/relevance to various differences in learners, the research project draws on the methodological framework of the ‘*Praxeological Sociology of Knowledge’* (Bohnsack, 2010). Providing a methodological framework as well as a meta-theory, this theory comprises a distinction between two levels of knowledge. On one hand, the *reflective or theoretical knowledge* is accessible by those under research and can be made explicit during communication. On the other hand, the *practical or incorporated knowledge* remains implicit. Since incorporated knowledge gives orientation to action, it can be assumed that shared orientations among teachers, which structure the processes of ‘doing difference’, can be identified.

Aiming to understand primary mathematics teachers’ everyday practice, its inherent logic and its underlying orientations while constructing differences as ir/relevant, the research investigates *how* mathematics teachers construct differences in learners as ir/relevant for mathematics learning. In particular, we try to detect underlying shared collective orientations (incorporated knowledge) and general patterns of orientation during the process of constructing differences. To answer these questions, primary schools’ mathematics teachers are invited to participate in group discussions, in which they are asked to share their experiences with teaching mathematics. Data provided by the group discussions is analysed using the Documentary Method (Bohnsack, 2010).

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MEASURING PROPORTIONAL REASONING IN GRADES 5 TO 7: WHAT DEVELOPS AND WHAT DOES NOT?

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Proportional reasoning is inevitably an important psychological construct with many structural connections to different fields of mathematics (Boyer and Levin, 2015). There are various views on the nature of the development of proportional reasoning, however, it clear task variables play a crucial role in affecting students' solution strategies on proportional reasoning tasks (Vanluydt et al., 2020). The current investigation aimed to reveal the role of task characteristics in students' performance in a proportional reasoning test designed for lower secondary age groups. The task characteristics varied were as follows, (1) pictorial versus numerical tasks, (2) pictorial tasks with either discrete or continuous quantities, (3) different types of the options in closed-question-format items.

In the current research, a system of on-line proportional reasoning tasks has been developed on the eDIA platform (Csapó & Molnár, 2019), and administered to 92 5^{th} grade, 87 6^{th} grade and 88 7^{th} students from the same schools. The reliability of the test was α=0,783.

Results suggest that there is an overall significant difference in performance between 5^{th} graders and the older groups. Refined analyses revealed that it was the textual format answer type that were much easier for the 6^{th} and 7^{th} graders. However, when the item stimulus was a continuous quantity with either discrete or continuous pictorial answer options, the differences between the age groups were not significant.

The educational implications involve a strong emphasis on the use of more diverse tasks in terms of both the item stem and the answer format from as early as primary school grades. Another important finding is that quantities and proportions depicted in pictorial format still seem to cause serious difficulties in grades 6 and 7, indicating the lack of variety in the task characteristics used in the mathematics classes.

ADDITIONAL INFORMATION

This work was supported by the Research Programme for Public Education Development of the Hungarian Academy of Sciences.

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PROGRESSION TOWARDS USE OF FORMAL MATHEMATICAL REGISTER: EXTENT OF STRUCTURE FRAMEWORK

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In this paper I present outcomes of the extent of structure for progression towards use of formal mathematical register. The extent of structure in this paper is seen as distinguishing quantities in learners' work as they emerged through the data. A key aspect for making progress in mathematics involves moving to increasingly more formal mathematical register and efficient calculation strategies. The South African evidence of poor performance and prevalance of use of inefficient strategies (Ensor, et al., 2009), tends to emphasise the need for moving on from concrete counting strategies, but does not provide ways of effectively realising this move.

This paper is located within a small-scale intervention that focused on exploring mathematising processes of South African Grade 2 isiZulu speaking learners. Data was collected from interviews and pre-and post-tests in two classrooms with isiZulu speaking learners, in which one class constituted an intervention group and the other a control group. In order to analyse the extent of structure in the broader study I developed a framework which comprised progressive indicators emerging from the data and existing framework from Ensor et al. 's (2009) paper.

The analytical framework to analyse the extent of structuring was developed and comprised indicators indicating progression. Progression ranged from 'No Overt Models with Incorrect answers' to 'No Overt Models with correct answers with the former being the least indictor. Furthermore, indicators were coupled with incorrect and correct answers since performance is a major aspect in mathematics. The results from the analysis showed that distinguished quantities within pictorial models were associated with more correct answers which suggests that similar interventions can assist learners to move towards a more formal mathematical register. The framework further provides ways of realise the move from concrete to more formal ways of solving mathematics.

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HOW 4TH-GRADERS WORK ON APP-BASED BALANCE SCALES TASKS

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Various difficulties in algebra and especially in dealing with equations in secondary school have been widely reported (e.g., Schliemann et al., 2006), prompting calls for Early Algebra introducing algebraic concepts beginning at the primary school age. Among other things, the balance scales model could have a lot of potential for this.

Balance scales tasks can be categorized regarding expected working processes, oriented to equation solving strategies as simplification or substitution, as well as in terms of task structure, mainly the number and combinations of unknowns (Bräuer, 2022). The present pilot study of the first author's PhD-project refers to four tasks of one (challenging) task structure, systematically varied with respect to suggested solving processes. Semi-structured individual interviews were conducted with a heterogeneous sample (in terms of mathematical achievement) of 9–10-year-old fourth graders (N=18). Considering our experiences with manual material in previous studies and the potentials in combination with eye-tracking in a planned upcoming study, a self-developed web app was used to present the tasks and allow a manipulation of digital material to solve them. This also made a screen recording (including sounds) possible to facilitate a detailed reconstruction of the students' solving processes.
Our research questions were: *Which strategies can be identified for balance scales tasks that vary regarding the suggested working procedures? How is the digital material of the web app used by the students?*

While a previous study (Bräuer, 2022) has already shown that students of this age are able to successfully work on "equations" or even "equation systems" represented as balance scales using algebraic strategies in an informal way, the presented study provided further important findings. Despite the increased task difficulty, different solving procedures matching task features could be reconstructed. Furthermore, features of the web app as, e.g., simultaneous removal of several elements, supported students' working processes. Concerning the analyses, the screen recording proved its potentials, which could be further strengthened in combination with eye tracking.

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EARLY DETECTION OF RISK FOR MATH LEARNING DIFFICULTIES BASED ON SYMBOLIC AND NON-SYMBOLIC TASKS

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Early detection of risk for math learning difficulties (MLDs) is important for timely intervention. But detection protocols in Chile are based on children's delay in attaining school math curriculum milestones, which may come too late for effective intervention. Moreover, difficulties with specific math contents may arise from multiple cognitive and non-cognitive causes, undermining their usefulness (Estévez et al., 2022; Jiménez et al., 2020). In working towards detection protocols based on basic math skills, a topic of interest is the relevance of performance in symbolic and non-symbolic numerical tasks. Schneider et al. (2017) reported that symbolic tasks associate more strongly with mathematical competence, and here we ask if this finding extends to the detection of MLDs. We developed a battery of basic math skill tasks and tested it with 725 children from 1st to 6th grade, with a 15% of prevalence of MLDs (as diagnosed by their schools). Here, we focus on two of these tasks: number comparison (symbolic) and numerosity comparison (non-symbolic). We used as indices, for both tasks and all children, their log-scaled median response time; our detection criterion was to have an index higher than the median index plus one mean absolute deviation (on a grade-per-grade basis). Results showed that both tasks flag similar numbers of children (21.0% vs. 20.6%), but children flagged by the symbolic task criterion were more likely to have an MLD (28.9% vs. 18.8%). The symbolic task advantage occurred across all school grades, supporting the need of using symbolic tasks for MLD detection.

Acknowledgments. This research was supported by the Chilean ANID funds FONDEF ID18I10002, Milenio NCS2021_014, and Basal FB0003.

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EXPLORING TEACHER'S SELF-REGULATED TEACHING IN MATHEMATICS: TAKING DISCOURSE ANALYSIS AS AN APPROACH

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Developing students' self-regulated learning (SRL) competencies has been an emerging issue for curricular reforms in many countries to address complicated social changes and sustainable development. Taiwan is no exception. Its latest released mathematics curriculum guidelines emphasize that SRL lays the foundation for lifelong learning and career development. It is noted that students' development of SRL needs their teachers' explicit guidance and teaching, i.e., self-regulated teaching (SRT) (Kramarski, 2018). However, current knowledge of how teachers conduct SRT is limited. Therefore, this study adapted Dignath and Büttner's (2018) conceptual framework to analyze SRT to fulfill the research gap. SRT consists of motivational, cognitive, and metacognitive dimensions in this conceptual framework. We constructed a coding scheme for discourse analysis within the framework. Six secondary school mathematics teachers participated in this study. Two lessons were consecutively videotaped when the teachers began teaching a seventh-grade math unit. After videotaping, participating teachers were interviewed to learn their SRT beliefs and practices. These data were transcribed verbatim for data analysis. Research findings suggested that a) Although reformed curricular documents encouraged SRL, teachers' SRT discourse occurred less in the classroom. In addition, the SRT discourse tends to be cognitive and metacognitive. The motivational discourse rarely occurred. b) Teachers who took a student-centered approach performed more metacognitive discourse than teachers who took a teacher-centered approach in conducting SRT. Evidence from the interview suggested that the SRT difference may be due to the teaching structure and teacher-student interactions.

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A LONG-TERM INVESTIGATION ON JUNIOR HIGH SCHOOL STUDENTS' PERSPECTIVES ON WHAT TEACHERS SHOULD DO TO ENHANCE THEIR MATH LEARNING MOTIVATION

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Despite their impressive mathematics achievements, the students in Taiwan have been consistently reported to have low mathematics learning motivation in TIMSS and PISA (Mullis et al., 2020). Thus, this study endeavors to investigate junior high school students' perspectives on what teachers should do to increase their learning motivation in mathematics.

Bearing the ideas of naturalistic inquiry, a year-long investigation was conducted on a class of 30 junior high school students through students' question-oriented mathematical diary writing. During the study, the students were asked a total of 198 open-ended questions across 68 diary entries. The initial questions on the list were general and broad, such as "My biggest gain in today's math class was..." The subsequent questions were generated based on the content analysis on the students' responses to the previous questions. Thus, the questions became increasingly targeted and focused. For example, several students responded to the question "What should I improve in today's class?" with answers like "I think [we] have to ask questions if we do not understand, but to be honest, [I] don't always have the courage to do so." Based on these responses, we generated the questions that were specifically related to students' asking questions, such as "In today's class, I bravely asked a question because…"The responses helped us understand the motivational factors affect students' willingness to engage in class with regard to asking questions. Based on the responses, we turned our attention to students' feelings to teacher moves and asked the questions such as "If the teacher say 'Have you ever thought about it first?' after Ming asked a question, what do you think Ming would feel?"

Through inductive analysis on the students' responses, 139 teacher moves to enhance students' motivation in 11 different learning activities (e.g., doing math homework, listening to the class, and participating in discussion) were identified. The motivational factors were also revealed, including *creating a secure learning environment, setting achievable goals, promoting student self-determination, expressing concern and encouragement, and displaying positive emotion as a teacher*.

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WHAT CAN A MOBILE SOFTWARE APPLICATION (APP) DO IN THE MATHEMATICS CLASSROOM?

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There have been many studies about the effects of games and information technology in mathematics education. Game-based learning, to no one's surprise, usually has positive effects on students in the affective domain, and some studies show that information technology improved the performances of average or lower-achieving students. In recent years, mobile apps, a combination of games and technology, have become a trend in our lives, and researchers naturally wish to explore whether they can help students' mathematics learning. In addition, prior studies seldom discuss the long-term effects of computer games in mathematics education, and concept retention is an important aspect of learning over time (e.g., Rohrer & Taylor, 2006). The idea of concept retention is also closely related to a group of teaching tools that the authors aimed to explore: concept-foundation modules, which are currently popular in Taiwan. This study aims to investigate the effects of concept-foundation module apps on students' retention of mathematical concepts. This report will introduce the authors' efforts in an exploratory study, where they collaborated with program developers and in-service teachers to create an app for a concept-foundation module, and examine its impact on elementary students' concept retention in mathematics learning.

We found that the app could motivate students and make mathematics classes more interesting. Cognitively speaking, students could learn by themselves, and the teachers were mission givers and order maintainers. We also found that there was no need for superfluous practices. Three times would create the best results. It was also found that this app could overcome the age and gender gaps, which could mean that it is easy to teach with apps in a co-ed and mixed-age class. There was even no difference among students of various performances. Apps are good tools for differentiated teaching. Apps can not only change the teaching of mathematics and the learning atmosphere of students, but also provide the opportunities to integrate technology into mathematics classes. An alarming result is that those who liked mathematics had significantly better scores in both the level of entertainment and education, and the level of performance and handling, showing that the motivating effects of apps for those who do not like mathematics would be less than the effects for those who like mathematics.

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EXPLORING THE KNOWLEDGE SOURCES OF MATH EDUCATION STUDENTS

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According to Shulman (1986), in addition to knowledge types, there are three primary sources of knowledge about teaching: disciplined empirical or philosophical inquiry, practical experience, and moral or ethical reasoning. These sources can help pre-service teachers (PSTs) develop strategic knowledge— “wisdom of practice,” that is necessary to navigate conflict situations in the classroom (Shulman, 2007). Unlike types of knowledge studies, the sources of knowledge in teacher education have not been sufficiently studied.

Linking theory to practice is a significant challenge in teacher education, as noted by Loughran and Hamilton (2016). Thus, our RQs were: What sources of knowledge are perceived as significant by PST students? What sources of knowledge do they use when working on their assignments, and how? What sources of knowledge do PST students intend to use in the future? We analysed data from ten PSTs: interviews, questionnaires at the start and end of the program, written assignments in a course on math education, field notes from various program courses, and conversations with program participants. We compared two populations of PSTs (undergraduates and second-career).

We found that PST students mentioned and used sources of knowledge based on disciplined empirical or philosophical inquiry related to the constructivist theory and were willing to use it in the future. They developed a connection between theoretical and practical sources of knowledge by applying theoretical principles in planning and carrying out a didactic intervention during the didactic courses. They expanded their point of view rooted in moral or ethical reasoning to a more general perspective. We conclude that exposure to general theories in education, followed by studying content knowledge courses for teaching mathematics that include applied elements, can be seen as a bridge connecting theory and practice. Practical experience in which students apply local theories in their teaching leads to consolidating strategic knowledge for teaching.

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SPATIAL REASONING IN GEOMETRY AND CARTOGRAPHY

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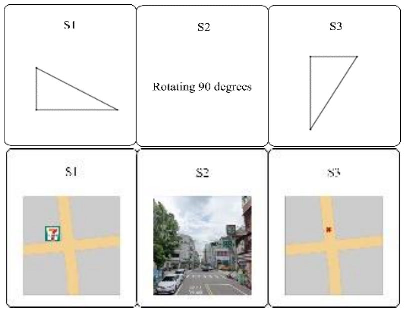

Spatial reasoning plays crucial role in both geometry and cartography. However, rare studies have compared similarities and differences in spatial reasoning between solving geometry and cartography problems. In this study, we designed two instruments, geometry-rotation test and map-reading test (see examples on right side), and adopted ERP methodology to examine brain activities specific to spatial reasoning. Spatial reasoning specific to geometry rotation refers to mental rotations occurred when one perceives geometric diagram, creates mental images of the diagram, and then performs rotations based on the created images. Spatial reasoning specific to the map reading refers to perspective transformation occurred when one identifies real-world landmarks and relationships from an egocentric reference frame, puts these cues together, and places oneself on the map via an allocentric reference frame so that one can determine the location relationship. Both instruments include three stages that allow to compare perception on the objects (geometry diagram vs. map) (Stage 1), spatial reasoning performance (mental rotations vs. perspective transformations) (Stage 2), and evaluating the answers (Stage 3). Each instrument has 320 trials. For both Stage 1 and Stage 2, analyses showed that students paid higher attention on perceiving a geometry object than perceiving a map because amplitude for P100 component identified in geometry-rotation test was greater than that in map-reading test. As Zacks et al. (1999) argued that perspective transformation often occurs near the parietal-temporal-occipital (PTO) area in brain, the Pz electrode was compared. We found that perspective transformations required higher mental efforts than mental rotations in Stage 2. The finding referred that much cognitive efforts was required in recognizing landmark in cross-road map, identifying signboard corresponding to landmark in reality of street-view, and then coordinating between cross-road map with landmark and signboard in street-view. Cognitive efforts for mental rotation were less when compared to perspective transformation. Additionally, Cz electrode was also examined as it involves motor imagery. The analysis indicated that perspective transformation created a higher amplitude in Cz electrode than mental rotation did. The result implies that taking perspective transformation is likely to act motor imagery but not mental rotation.

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MATHEMATICAL MODELLING AND CODING OF SOCIAL JUSTICE ISSUES: SUPPORTING STEM IDENTITY

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Coding mathematical models of real-world social justice issues is an agentic learning strategy that supports STEM identity development for minoritized undergraduate students in STEM (Ruttenberg-Rozen et al., 2022). Many underrepresented people enter STEM education to make an impact on their communities (Carlone & Johnson, 2007). Subsequently, the activity of using digital tools to model the mathematics in social issues helps learners to see the relevance of mathematics to their interests (Rubel et al., 2016), all the while building digital skills. These experiences can promote a sense of belonging in STEM and STEM identity (Ruttenberg-Rozen et al., 2022).

In our study, we used Carlone and Johnson's (2007) framework of STEM identity to explore the questions: How does the process of mathematically modelling and digitally coding social justice issues support the development of a STEM identity? What new relationships do learners see between mathematics and social justice issues? STEM identity refers to how an individual sees themselves as a STEM person. It outlines how the satisfaction of an individual's need for competence, performance, and recognition contributes to their sense of STEM identity (Carlone & Johnson, 2007).

Our study occurred over the course of one year. We recruited six undergraduate women in STEM to participate weekly in mathematical modelling and coding activities of social justice issues. We used ethnography, semi-structured interviews, artefacts, and field notes to gain insights into our research questions. In this poster-presentation, we share narratives from our participants regarding the impact of the mathematical modelling and coding activities on their STEM identity. For example, one participant concluded, "Working on this project has truly changed our perspective surrounding our own identities in the STEM space."

ACKNOWLEDGMENTS

We acknowledge the generous funding of this research by the Social Sciences and Humanities Research Council of Canada.

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PROMOTING CREATIVE THINKING IN SCHOOLCHILDREN WITH APPLIED PROBLEMS IN MATHEMATICS

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Current approaches in mathematics education promote the teaching of creative thinking (CT) to develop a deep conceptual understanding of mathematics, and many nations are including explicit CT learning goals in their curriculum (Hadar & Tirosh, 2019). The current education programs in mathematics are at that stage of development where they actively cover real-life tasks and scenarios.

The present study aims to investigate the effects of applied mathematical problems on the creative thinking ability of schoolchildren.

The CT assessment procedure involved testing respondents before and after the applied problems in algebra and geometry using the Williams method. The study period covers the first semester of the 2021 school year. The study sample consists of 7th graders attending the secondary school in one of the central cities in Israel. Overall, there were 32 students enrolled in class. The study was conducted as a step-by-step analysis of the children's level of CT before and after the active introduction of practical mathematics tasks into their curriculum. According to the current timetable of all 7^{th} graders, students attended 3 algebra lessons and 2 geometry lessons every week. In doing so, teachers promoted the use of applied tasks in these subjects. The applied problem solving analysed in the current study was applied in one lesson of algebra and geometry every week.

The analysis of research results provides evidence supporting the expected dependence. In the baseline tests, for instance, the majority of respondents exhibited an average level of potential for creativity (15 children). Meantime, respondents with high and low levels of CT made up smaller groups. The subsequent exposure to applied mathematical problems for everyday situations took place systematically throughout the course. The second creativity test demonstrated an improvement in the creative thinking ability among respondents.

Particularly, most children had better creativity skills and higher levels of creative thinking (19 out of 32). The present findings may help improve the education process in exact sciences (such as mathematics) to enhance the role of creative thinking in schoolchildren's further life.

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ALGORITHM IN MIDDLE SCHOOL MATHEMATICS TESTBOOKS

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Algorithmic thinking which has become an increasingly common concept in the mathematics education literature in recent years is defined as the ability to understand, apply, evaluate and produce algorithms. Algorithm is the logical and sequential expression of the path, method and process to be followed during problem-solving within the framework of certain rules (Michael & Omoloye, 2014). For the development of algorithmic thinking skills, in some countries, algorithm teaching is limited to informatics courses (e.g. Türkiye, Ukraine), while in other countries, algorithms are included in mathematics programs as well as informatics courses (e.g. France).

In Türkiye, although algorithms and algorithmic thinking are not included in the mathematics curriculum, it is seen that questions involving algorithms (QA) are included in the middle school skill-based tests book published by MEB (Ministry of National Education). The purpose of this study is to examine the QA in these mathematics workbooks. In the study, firstly, the distribution of the QA at the 5th, 6th, 7th and 8th-grade levels of middle school, and secondly, the algorithm structures contained in the QA were examined.

According to the findings, it was determined that there were equal numbers of QA at the 5th and 6th-grade levels, 7th and 8th-grade levels, and the number of questions was more in the 7th and 8th grades. When the subject distribution of QA, it is seen that there is a cluster in the field of numbers, while at least one question in algebra and geometry at each level contains algorithms. When we look at the algorithm structures in the QA, in one group, the operations to be performed are given step by step, while in another group, the progress steps for the solution are given according to the necessary conditions. The results of the study show that although the outcomes for algorithmic thinking skills in mathematics courses are not explicitly included in the institutional programs, they are taken into consideration in practice. In future studies, it is aimed to explicitly reveal the implicit presence of algorithms in the Turkish mathematics program and teaching practices and to reveal the development and effects of algorithmic thinking skills through skill-based questions in terms of both teachers' implementation processes and students' learning processes.

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INTEREST DEVELOPMENT WHEN MODELLING DISEASES

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Simulation science has been more present in the media since COVID-19, even interactively engaging the media, such as in an interactive Washington Post article on "Flatten the Curve" that went viral (Stevens, 2020). Research indicated that computer-based science learning positively impacts student attitudes (Hillmayr et al., 2020). Such simulations of actual science can hence provide new opportunities to draw students' interests, making us wonder '*how can modern mathematics and current science topics in mathematics classes influences learners' interest in the subject mathematics?'*.

We follow the four-phase model of interest development (Hidi & Renninger, 2006) to characterize stages of interest, with our focus set on situational interest – the phases '*catch*' and '*hold*'. By analysing responses to a survey, we empirically interrogate whether the topicality of the subject as well as the novelty of the mathematical method are suitable for triggering situational interest. Furthermore, we measure whether the active involvement of learners in modelling and analysing the results as well as the social relevance of the learning content are suitable as influencing factors in order to maintain situational interest. The mentioned simulations are a in time and space ODE-model with collision, similar to the famous Boltzmann equation. An alternative which is also widely used are cellular automata. They are discrete in time and space and are based on recursive sequences, which are easily implementable in e.g., Excel and makes them accessible in many curricula. Regarding interest structure, the mathematical project itself appeals to a variety of types according to the RIASEC+N model (Dierks et al., 2016) due to its variety and novelty. A special focus is on networking, realized due a permanent exchange during modelling, implementation and analysis.

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CONSTRUCTIVELY ALIGNED TEACHING FRAMEWORK: A CASE FOR MATHEMATICAL TEACHING FRAMEWORK

Lizeka Gcasamba

Higher education is increasingly emphasizing two schools of thought: theory and practice. The first comes from the instructional design literature, while the second is based on constructivist learning theory. Constructivism emphasizes the importance of the learner's activities in producing meaning. On their end, instructional designers have emphasized the importance of alignment between the learning objectives of a course or unit and the targets for assessing student performance. Constructive Alignment (CA) is a teaching principle that combines constructivism- the idea that learners construct or create meaning out of learning activities and what they learn, and an instructional design concept that emphasizes the importance of defining and achieving intended learning outcomes (Biggs and Tang, 2011). Such a teaching principle is assumed in Adler's (2021) mathematics teaching framework (MTF). The MTF tool was conceptualised by Adler (2021) as a teaching tool that emphasises the coherence between the teaching and learning activities with the object of learning. There are strong commonalities between MTF and CA. In this poster I describe a framework for designing instruction, instructional mathematics teaching framework (IMTF). Most importantly, the aim of this poster is to argue that the IMTF is a framework that foregrounds a notion of constructive alignment.

The CA is a principle that is characterized by three elements, namely, (1) intended learning outcomes, (2) the teaching and learning activities, and (3) the assessment. On the other hand, the IMTF is characterised by three interacting components in the teaching of a mathematics lesson that are constructively aligned. The first step relates to the articulation of what needs to be learned—the object of learning in a lesson. This step is then followed by the second step which concerns the connection of what is critical to learn through teaching and learning activities such as, exemplification, explanatory talk and learner participation. The last step is the consideration of assessments that are aligned with the identified object of learning.

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MEASURING EXECUTIVE FUNCTIONS IN GENERAL AND SPECIFIC MIDDLE-SCHOOL LEVEL GEOMETRY CONTEXT

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Executive functions (EFs) are a set of cognitive abilities that allow us to function properly in our daily lives (Miyake & Friedman, 2012). EFs are also critical to formal learning and especially mathematics. For learning geometry, the relationship was only tested with working memory and intelligence. No studies have investigated the involvement of other EFs. The studies that assess EF generally use cognitive tasks in a general context (e.g., color-word Stroop task to measure inhibition). No study evaluated EFs in the context of middle-school level geometry. This study aims to examine two questions:

- What are the characteristics of the relationship between basic EFs (working memory (WM), shifting, inhibition) as they are measured in a general laboratory context and success in a geometry test?
- What are the characteristics of the relationship between basic EFs in a general context and the same EFs measured in a geometric context?

The study included 106 adults aged 18 to 35. They solved middle school-level geometry problems devised by us (EFGT). Each problem required mostly inhibition, working memory, or task switching. Next, they performed EF tasks in a general context. The results revealed a correlation between accuracy in the EFGT task and inhibition and spatial WM. We also found specific correlations in inhibition and WM between the same EFs measured in specific and general- contexts. The study demonstrated the importance of EFs to middle-school geometry and the differences and similarities between studying EFs in general and in specific contexts. We discuss the advantages and limitations of the EFGT tool that measures EFs in a geometric context.

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A STUDY OF THE IMPACT OF MATHEMATICS TEACHER ON THE VALUE FORMATION OF FIRST GRADE JUNIOR HIGH SCHOOL STUDENTS

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This study relies on the international comparative survey "The Third Wave" conducted by Seah et al. (2012) to investigate the formation process of students' values in mathematics education in Japan and the influence of mathematics teachers' values on it. While previous studies have explored how mathematics teachers in various countries handle diverse values for quality learning, using WIFItoo as the main research tool, a case study by Kinone et al. (2020) in Japan revealed the diversity of student values and the similarities and differences between teachers and students. In Japanese mathematics education, it has been noted that both teachers and students place a strong emphasis on "others' explanations," while other aspects such as "talent," "rigor," "facts in the mathematical world," "real-world applications," and "creativity" are not given as much importance.

However, the formation process of students' values and the influence of mathematics teachers on them have not been adequately studied, and this remains an issue for future research. Building on the results of previous studies, this research aims to investigate and analyze the formation process of students' values in mathematics education in Japan and the influence of mathematics teachers on them by examining what kind of values students receive from their mathematics teachers during one year of secondary education.

In Japan, there is a phenomenon called the "Secondary Grade 1 gap," in which students' emotional aspect to mathematics decreases significantly when they transition from primary to secondary education. It is thought that the value system of mathematics teachers' teaching due to the subject-teacher system starting from secondary education has an impact. This study aims to clarify the changes in first-year students' values towards mathematics education at the beginning and end of the year through a questionnaire survey, compare them with the values of mathematics teachers, analyze their impact. The results indicated that at the end of the year, the teacher's values of explaining to others had influenced the classroom. It was also evident that the value gap between teachers and students had been resolved.

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THE PROFESSIONAL DEVELOPMENT OF ELEMENTARY TEACHERS OF MATHEMATICS TEACHING BY CO-LEARNING INQUIRY PROCESS

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There is always room for improving teachers' teaching practice. In the past, several researches investigate teachers' professional development (PD) focus on mathematical tasks design (Zaslavsky, 2007), contents analysis of textbook and discussion of teaching implementation (Lin, 2007). Regardless of research focus researchers adopted, the processes of co-learning inquiry between researchers and teachers which focus on teaching practice plays a critical role (Jaworski, 2008). But what's the core of mathematics teaching practice? The mathematical problem is the basic unit of curriculum contents and the focus of teaching and learning. Based on researchers' argument on teachers' PD improvement and the feature of mathematics teaching and learning, the Co-Learning Inquiry Community (CLIC) was made up by mathematics educator and 10 school teachers. From the year of 2020 to 2021, case study was adopted as method and 10 times CLIC meeting that focus on non-routine mathematical problems design and discussion were conducted, 2 times interview and 3 lessons observation from each teacher were collected. The results from data analysis indicates that teachers have abilities on non-routine problems design and transformation. Fifty non-routine problems were designed through CLIC discussion. The teaching perspectives of teachers was transformed from teacher-centred to student-centred after research participation. When teachers implemented non-routine problems they design in classroom, they provided more questioning to guide students' thinking, and more mathematical discussion and conversation happened among students. The evidences from different data we concluded that adopted the approach of co-learning inquiry between educator and teachers with focus on problem design could improve teacher's PD on mathematics teaching, and make students' mathematics learning with thinking and understanding in the inquiry processes.

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REVISITING TPACK: A DIALOGUE BETWEEN PEDAGOGY AND TECHNOLOGY - AN EMPHASIS ON RATIO AND PROPORTION

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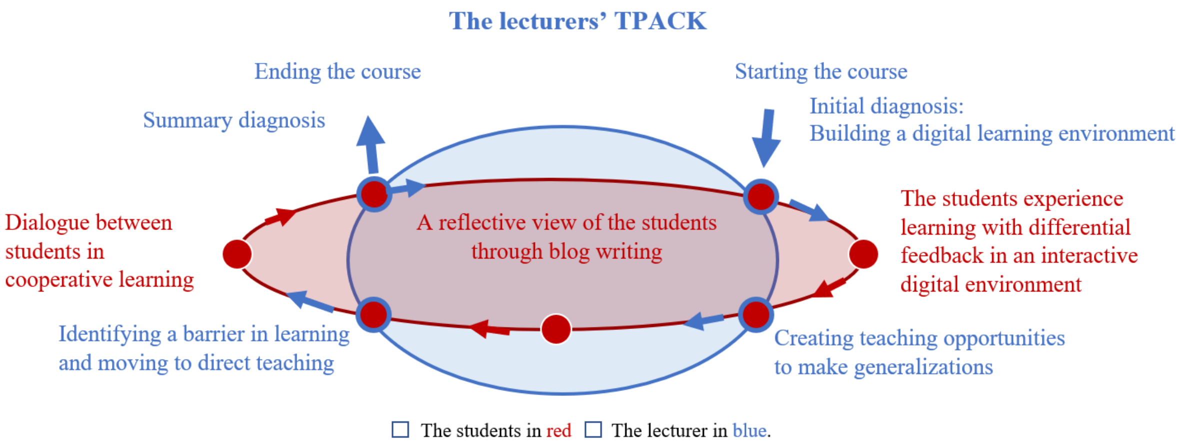
The study accompanied an online course for in-practice elementary school teachers, focusing on the impact of independent learning processes on students' proportional thinking and problem-solving strategies. The course involved interactive computerized activities and follow-up discussions, aiming to allow students' intuitive knowledge to transform into formal mathematical knowledge. The proposed model focuses on seven junctures in which the lecturer reflected on her teaching based on student feedback and describe the dialogue between pedagogy and technology. Our model builds on Cohen's (2016) model of teaching and learning and based on Zaslavsky and Leikin (2004). The model refers to aspects of TPACK according to Koehler et al., (2013).



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SPANISH AND GERMAN PRE-SERVICE TEACHER'S ANALYSES OF A TEACHING-LEARNING SITUATION ON FRACTION OPERATIONS

Pedro Ivars[1], Sebastian Kuntze[2], Jens Krummenauer[2],

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Teacher noticing has been identified as a knowledge-based reasoning competence (Sherin et al., 2011). Therefore, the knowledge used by pre-service teachers when analysing a teaching-learning situation plays an important role in the further development of this competence. From a mathematics education point of view, teacher noticing should not neglect mathematics-related aspects of teaching-learning situations, as it would be the case if teachers exclusively draw on pedagogical knowledge (PK) and not on pedagogical content knowledge (PCK). Still, little is known empirically about the roles of and relationship between PK and PCK when pre-service teachers analyse a situation and possible teacher reactions. Consequently, this study aims at exploring the research question: *What are the roles of and the relationship between PK and PCK in pre-service teachers' analyses of a teaching-learning situation?* In particular, the identification of cases in which PK-focused analyses might have hindered the pre-service teachers in drawing from relevant PCK for analysing the situation and possible teacher reactions is of great interest. 36 pre-service teachers (PTs) from a Spanish University and 56 PTs from a German University were asked to analyse a teaching-learning situation related to fraction operations. First, they had to analyse students' understanding considering the strategy used and then they had to complete the situation providing a teaching decision on the basis of students' understanding. PTs' written answers were analysed using an interpretative coding. The results indicate that PCK and PK aspects within the pre-service teachers' knowledge-based reasoning are frequently in a complex interplay. In particular, cases in which an incomplete analysis of the situation coincided with a strong PK focus suggest that a metacognitive support to these pre-service teachers focused in the significance of a specific mathematics-related analysis focus could help them to productively combine PK and PCK foci when analysing teaching-learning situations.

ACKNOWLEDGEMENTS

The project coReflect@maths (2019-1-DE01-KA203-004947) is co-funded by the Erasmus+ Programme of the European Union. The European Commission's support for the production of this publication does not constitute an endorsement of the contents, which reflect the views only of the authors, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

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A NUMBER TALK QUESTIONING FRAMEWORK FOR ADVANCING RESEARCH AND TEACHER DEVELOPMENT

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Number Talks are a popular routine in K-12 classrooms; however, there is little research that focuses on teachers' implementation (Matney et al., 2020). Number Talks are intended to be a 5- to 15-minute classroom discussion around a computational problem or sequence of related computational problems. Teachers' questions during a Number Talk are central to providing opportunities for students to share and justify their mathematical thinking and consider the thinking of others.

Our Number Talk Questioning Framework was developed from analysis of 30 K-12 teachers' Number Talks. Based on the Number Talk recommendations (Humphreys & Parker, 2015; Parrish, 2010), we identified seven types of questions teachers can ask. The Number Talk Questioning Framework also captures opportunities that questions allow for students to engage with the thinking of others and instances where the teacher inserts their own ideas or language into the conversation are captured. Our NTQF draws from existing questioning literature but is unique in both the question-types found and the structure of the framework. Further, the framework serves dual purposes for both contributing to the emerging research in Number Talk s, and for teacher development of both Number Talk -specific and math classroom-general questioning practices. Our findings from application of the Number Talk Questioning Framework include descriptions of common questioning patterns teachers use and specific difficulties or challenges teachers have with questioning to meet the goals of a Number Talk.

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THE EFFECT OF MATH CLINIC USING K-UTF PROGRAM TO REDUCE MATH ANXIETY

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Purpose: It is necessary to develop a program that can cultivate the ability to autonomously manage math anxiety for the students who gave up mathematical learning due to math anxiety (MA).

Background: Tobias (1978)'s MA clinic at Wesleyan University is widely known as a program that applies psychological and non-psychological treatment at the same time, being called a complex treatment (Choi-Koh & Ryu, 2019).

Development program: A K-UTF (Korean Understanding-Treatment-Feedback) program was developed using the approach of a complex treatment and was applied to 26 students from a regular high school in order to find out its effect.

Table 1. K-UTF program and process

STEP	U (Understanding)	T (Treatment)	F (Feedback)
Title	I am anxious about ( ).	My Plan	Feedback from Teacher
Main Question	How anxious are you?	Know how I feel after encountering a problem.	Praise the students for a study well done and encourage him or her continue.

CONCLUSION

The K-UTF program reduced anxiety caused by math itself and learning strategies factors among mathematical anxiety factors, and also showed cognitive improvement effect on algebra and function. Through the self-assessment at an interview after the camp, the students expressed these feelings without forcibly excluding them when they felt MA again, and showed their will to overcome the moment by recalling the planning method and the positive emotions that they learned how to deal with. That is, even though MA did not disappear at once, it could be lessened through the strategies they learned in the treatment.

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REFLECTING ON PRE-SERVICE TEACHERS' OBSERVATIONS FROM SCHOOL INTERNSHIPS BASED ON REPRESENTATIONS OF PRACTICE

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School internships integrated in pre-service teachers' university education mostly aim at providing teachers with opportunities to connect classroom practice with theoretical knowledge from university courses. However, making classroom situations accessible for university seminar work which accompanies the internships is not easy, which complicates an effective reflection in the seminar group. Representations of practice (Buchbinder & Kuntze, 2018), in particular in cartoon format, could provide a solution: Using the DIVER Create tool developed in the project coReflect@maths, pre-service teachers can create classroom cartoons, so that classroom situations observed and represented by the pre-service teachers can subsequently be reflected on collaboratively. As research into collaborative reflection on self-developed cartoon representations of practice is scarce, an exploratory pilot study was carried out, aiming at exploring whether and how self-developed classroom situation cartoons can provide fruitful opportunities for collaborative reflection during internship phases. In order to focus reflection both on students' thinking and on teacher reactions, pre-service teachers were asked to represent classroom situations in which student mistakes occurred (e.g. Oser & Spychiger, 2005). First interpretive analyses of the pre-service teachers' representations of practice indicate that self-developed classroom cartoons can promote pre-service mathematics teachers' reflection during internships.

ACKNOWLEDGEMENTS

The project coReflect@maths (2019-1-DE01-KA203-004947, applicants: S. Kuntze, M. Friesen, K. Skilling, L. Healy, C. Fernandez, P. Ivars, S. Llinares, L. Samková), was co-funded by the Erasmus+ Programme of the EU. The European Commission's support for the production of this publication does not constitute an endorsement of the contents, which reflect the views only of the authors, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

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PRE-SERVICE TEACHERS' VIEWS ON INTERACTION IN THE MATHEMATICS CLASSROOM AS REFLECTED IN TEACHER-DESIGNED CLASSROOM CARTOONS

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In approaches to teacher noticing (e.g. Sherin et al, 2011), teachers' views (e.g. Pajares, 1992; Kuntze, 2012) might trigger what observation teachers focus on in classroom situations. However, in most vignette-based designs, the given classroom situations might influence the ways how teachers' views trigger their noticing. Inverting this vignette-based methodology so that teachers imagine or create classroom situations on their own promises that such individually-designed representations of imagined classroom situations reveal facets of teachers' views on classroom interaction. Through the use of digital tools which support representing classroom situations, such an inverted methodology is possible. This study thus explores pre-service teachers' views on interaction in the mathematics classroom through their individual cartoon vignette productions, on the base of the DIVER Create tool developed in the project coReflect@maths (www.coreflect.eu). We used a bottom-up interpretive approach to describe and condense potential evidence from the teachers' productions of cartoon vignettes to infer to their views on interaction. The results provide insight into how teachers imagined interaction features related to individual learning support, in particular. The insightful findings call for follow-up research with larger samples and specific instructional framing for the vignette creation, so that aspects of teachers' views on interaction can be targeted among other views on the mathematics classroom.

ACKNOWLEDGEMENTS

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PROSPECTIVE TEACHERS' REFLECTIONS ON THE INCLUSION OF MATHEMATICAL MODELLING IN TWO TRANSITIONAL TEACHING CONTEXTS

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We report the results of a research conducted during two academic years in a professionalising master's programme for secondary and baccalaureate education mathematics teachers in the Spanish context. The objective was to analyse the prospective teachers' reflections on the inclusion of mathematical modelling during their educational internship experiences, which were developed in two transitional teaching contexts (from face-to-face to virtual and vice versa) due to the COVID-19 pandemic. We analysed these reflections with the Didactic Suitability Criteria (DSC), which was the same tool used by the prospective teachers to guide the reflection on their own educational practise. The DSC are a theoretical tool to assess mathematical teaching and learning processes (Breda et al., 2017). The DSC are structured in a guideline of six criteria (focused on a specific aspect of the educational process), and each of them has its respective components and indicators that allow its operationalisation. In our study, the DSC allowed us to identify the aspects that the prospective teachers prioritised when they reflected on modelling.

We followed a qualitative research methodology from an interpretative paradigm, which consists of a content analysis on 236 Master's Degree Final Projects submitted by the prospective teachers during the 2019–2020 (face-to-face to virtual context) and 2020–2021 (virtual to face-to-face context) academic years. As a main result, we highlight that, in both courses, the «epistemic» and «ecological» aspects were prioritised, while the «interactional» and «mediational» aspects were scarcely considered in the prospective teachers' reflections on the inclusion of modelling during their educational internship experiences.

ACKNOWLEDGEMENTS

This research was conducted within the Project no. 72200458 funded by ANID/PFCHA (Chile); and Grant PID2021-127104NB-I00 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe".

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THE CHARACTERISTICS OF MATHEMATICAL MODELLING IN THE INQUIRY-BASED CLASSES

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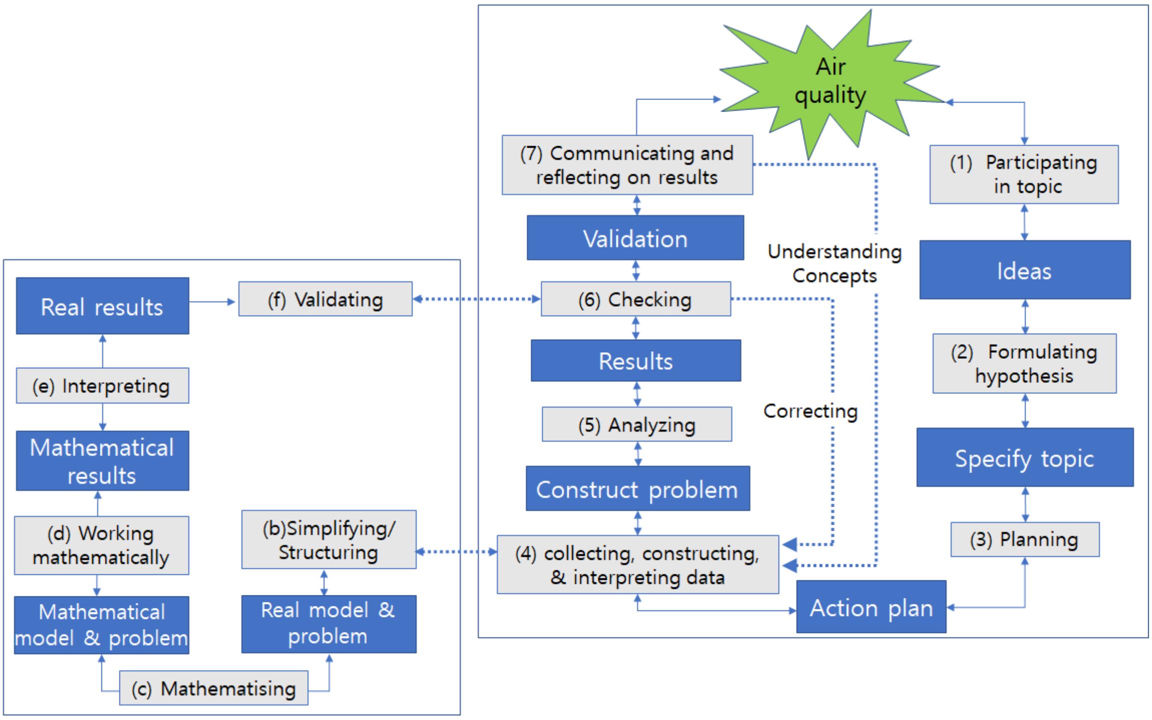
Through the integrated classes of mathematical inquiry and mathematical modelling processes, the researcher confirmed the relationship between them and presented the characteristics of mathematical modelling. The researcher used the integrated model presented by Sala, Barquero, and Font (2021) to analyze the features of mathematical modeling in the inquiry classes. As a case study, the participants consisted of two groups of high school students. The data collected were recordings of the conversation and the activity sheets during 18 unites.

Figure 1: Relationship between the modelling and inquiry derived from this study



As a result, the inquiry and modelling are complementary. Mathematical modelling activities were embodied when the specific topic of inquiry were formed and the data was collected. Mathematical modelling process had accurate the conclusion of the inquiry. The inquiry could be solved with an inductive approach, but mathematical modeling required a deductive approach.

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CODING IN MATHEMATICS CLASSROOM AND STUDENTS' AFFECTIVE ENGAGEMENT

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Affective engagement in mathematics, which is referred to as the situational affective state individuals enter during the teaching and learning of mathematics concepts, has been found to be an important component of students' school experience because of its relationship to their mathematical performance (Wang & Degol, 2014). Lee et al. (2019) categorized the affective mathematics education into four factors: attitude, emotion, self-acknowledgement, and value). Using this as a theoretical framework, we investigated the effect of coding-implemented mathematics learning on students' affective engagement. In particular, the research question is: Does the coding-mathematics STEAM activity effect the development of high school students' affective engagement in mathematics?

The participants were 86 high school students (39 females and 47 males) who were enrolled in a two-week STEM summer camp. The students were entering grades 9th-12th in the upcoming academic year, and none of them had learned or used coding prior to the intervention. The lessons were taught on a daily basis over the course of 10 lessons and were designed to be more challenging as the intervention progressed. We implemented MAME (Lee et al., 2019) as a pre-test prior to the intervention and as a post-test after the intervention period.

The results revealed that students' affective engagement in mathematics was developed through participating in the coding-mathematics STEM activity. The paired sample t-test result showed that the mean difference of pretest and posttest was statistically significant ($t=10.532$, $df=85$, $p<.001$), and the Cohen's d effect size of mean difference was 2.29. We also analyzed the data for each factor of affective mathematics engagement (i.e., attitude, emotion, self-acknowledgement, and value). More detailed results will be provided in the poster presentation.

The findings leave educators and researchers alike with significant factors to consider regarding the future of students' STEM education. In particular, because the coding-mathematics activities were shown to be effective in influencing students' engagement, it is critical to provide mathematics instructors with sufficient professional development training in coding and with effective strategies for implementing coding into existing STEM-related courses and activities.

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LATENT CLASS AND PROFILE ANALYSIS ON PRIMARY SCHOOL STUDENTS' MATHEMATICAL MINDSET

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The purpose of this study is to develop mathematical mindset scale for primary school students and adopt latent class model (LCM) to show its profile analysis. The theoretical foundation is originated from Carol Dweck, who provides concepts of mindset. Mindset is considered an established set of attitudes or values of an individual. Carol Dweck indicates that individual's abilities or talents serve as implications for either a fixed mindset or a growth mindset. People with a fixed mindset holds the beliefs that abilities or talents are static and little can be improved, whereas those with a growth mindset believe that abilities or talents are dynamic and can be developed through effort. According to Jo Boaler, mathematical mindset influences learning greatly. However, little is known about children's characteristics of mathematical mindset. Firstly, this study develops mathematical mindset scale for primary school students. Secondly, this study adopts latent class model to show students' latent features and profiles as to mathematical mindset. The sample includes 930 pupils, who are fifth and sixth graders from Taiwan. Results show that Cronbach α coefficient of internal consistency is acceptable. Factor analysis has also confirmed the two factors of growth mindset and fixed mindset. In latent class model, three indicators, which are AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and CAIC (Consistent Akaike Information Criterion), indicate there are six latent classes. These six classes are as follows. (1) class I: low fixed mindset and high growth mindset; (2) class II: medium fixed mindset and high growth mindset; (3) class III: low fixed mindset and medium growth mindset; (4) class IV: medium-low fixed mindset and medium-high growth mindset; (5) class V: medium-high fixed mindset and medium-low growth mindset; (6) class VI: low fixed mindset and low growth mindset. Based on profile analysis, there are more male students in class I than female students, while there are more female students in class IV than male students. There are more fifth graders in class II than sixth graders. On the contrary, there are more sixth graders in class III than fifth graders. MANOVA (multivariate analysis of variance) also reveals that there are significant mean differences in mathematics mindset as to gender and grade. There will be prospective research conducted on possible reasons to influence the performance of mathematical mindset.

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CLASSROOM PROFILES OF LEVEL AND HETEROGENEITY IN STUDENT-REPORTED INSTRUCTIONAL QUALITY

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LMU Munich

Different approaches to measure instructional quality have been debated, including student and teacher ratings. Student ratings are viewed controversially (Senden et al., 2022): Some authors doubt that students have the competence to rate teaching quality, while others conclude that class-averaged student ratings provide reliable and valid measures due to students' daily experience with different teaching styles. Previous studies mainly use the average of students' ratings from each class. Beyond this, high within-class heterogeneity of students' ratings might indicate that instruction caters some, but not all students in a class. Therefore, we investigated profiles of student-reported instructional quality in mathematics classrooms not only based on the average level of students' ratings but also their within-class variability. Using a class-centered clustering approach, we intend to detect meaningful configurations of level and heterogeneity of student-reported instructional quality in terms of the Three Basic Dimensions (TBD) Classroom Management, Cognitive Activation and Student Support (Charalambous & Praetorius, 2018). Our preliminary dataset contains grade eight students' ratings of instructional quality based on TBD from $N = 112$ classes. Using Latent Profile Analysis, we identified four different classroom profiles. First evaluations reveal that a small profile with the consistently lowest average ratings shows consistently high heterogeneity. Contrary, a larger profile with highest cognitive activation showed consistently lowest heterogeneity. The results indicate that taking rating heterogeneity into account offers interesting insights into between-class differences in instructional quality. We further aim to relate these results to teachers' ratings of instructional quality, and to students' learning gain. It will also be of interest to study how class-average ratings of instructional quality (as estimation of "real" instructional quality) and students' individual ratings (as measures of students' perception of instructional quality) predict class-level and student-level learning.

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PATHWAYS OF ATTENTION: TECHNOLOGY AND TENSIONS

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We examine shifts in attention (Mason, 1998) of a prospective teacher, Lilac (pseudonym), that emerged from her engagement with a computational modeling task focused on issues related to climate change. Research conducted with teachers who incorporate social justice issues into their mathematics teaching has highlighted tensions related to navigating between the social issue and the mathematical content (e.g., Bartell, 2013). Shifts in attention can be provoked by experiences of tension or disruption (Mason, 1998), and in this presentation we consider the research question: How can we understand the role of technology in influencing a prospective teacher's attention during her exploration of interdisciplinary connections related to modelling climate change issues with mathematics?

We analyse Lilac's engagement with a task focused on collecting and modelling climate-related data using the micro-controller, the BBC micro:bit©. The micro:bit is an electronic device that attaches to a computer and includes built-in and external sensors that allow users to collect and analyse real-world data such as temperature and soil moisture. Our objective for exploring climate issues through coding with micro:bit was to leverage the technology's capacity to collect and model real-world data. We sought to provide accessible opportunities through coding to develop rich understandings and experience new pedagogies (e.g., Gadanidis, 2015), while also demonstrating best-practices for incorporating coding into mathematics learning (Gleasman & Kim, 2020). We found that Lilac's pathway of attention was influenced by the physicality of the technology and tensions around the use of electronics, and these tensions diverted her from the original purposes of the task. We offer recommendations for instruction and research in teacher education.

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DATAVIZ DESIGN: A STUDY OF INTENTIONS

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Incorporating social justice issues into mathematics teaching is fraught with tensions related to navigating pedagogical goals for the social issue and for the mathematics (Bartell, 2013). Gutstein's (2006) framework for teaching mathematics for social justice (TMSJ) frames such goals as two sides of the same coin. Key examples of this framework in action focus on how to critique, interpret, or infer from real-world data, such as unemployment, incarceration rates, and globalization (Gutstein & Peterson, 2006). Data visualizations are commonly used to convey information about social issues and there is a need to foster critical engagement with them since "data visualizations cannot be taken at face value or assumed to be neutral carriers of information" (Rubel, et al., 2021, p.252).

This study examines the task design intentions (Kieran, 2019) of (n = 16) preservice secondary mathematics teachers (PSTs) who developed tasks wherein they created data visualizations (dataviz) about social justice issues of their choosing. Our study responds to the following research questions: What visual data representations did PSTs incorporate into their infographics? What were the pedagogical goals behind their dataviz design decisions? Our analyses suggest that tensions emerged with respect to PSTs intentions for (i) how to capture and convey a societal issue with data, (ii) how to leverage technology to create interactive data visualizations, (iv) how to balance attention between the context and the mathematics, and (iii) what reasoning is used to interpret the dataviz. This study forges new ground by integrating Kieran's (2019) framework for task design research with Gutstein's (2006) TMSJ, and offers new ideas for how to support preservice teachers' abilities to plan for and implement TMSJ.

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INVESTIGATING STRATEGIES FOR SECONDARY MATHEMATICS TEACHERS LEARNING ON FORMATIVE ASSESSMENT

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Formative assessment (FA) is described as a pedagogical approach that underlines the importance of teachers using students' understanding to inform instruction and promote learning (Black & Wiliam, 2009). Looney et al. (2018) offered the 'Teacher Assessment Identity' (TAI) construct to conceptualize teachers' identity development in their role of assessor. Their framework includes five interconnected aspects: (1) *I know* (2) *My role* (3) *I believe* (4) *I am confident* (5) *I feel*. In this study, we look for evidence of these aspects in the cognitive and affective dimensions of the teachers' experience of FA practices. We ask: *How do secondary-school mathematics teachers perceive,* both cognitively and affectively*, their experience of FA processes for learning to assess their students?* Thirty-four teachers participated in two cycles comprised of: sourcing a rich mathematics task and constructing an assessment rubric for it; implementing the task in their own classrooms; collecting and assessing students' responses; sharing and receiving peer feedback; providing feedback to students; and reflecting on their experience. After completion of the cycles, teachers were given a reflective questionnaire focusing on cognitive and affective dimensions of their experience of FA practices in ways related to Looney et al.'s framework. Data analysis combined *direct analysis* (by looking for references in teachers' utterances to Looney et al.'s categories) and *inductive analysis* to allow sub-categories to emerge. Analysis revealed that the teachers emphasized their knowledge and awareness of various aspects related to planning and implementing FA in the classroom. We also found cases of teachers who felt they had the appropriate knowledge related to FA, but in practice, during the group activity, or while working in the classroom, they discovered that this practice is more complex than they had originally thought. The study's findings contributes to the broadening knowledge of teachers' perceptions regarding FA. It also provides insights into the opportunities and limitations regarding assessment cycles as a way to promote teachers' learning of FA.

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INTEGRATING EDUCATION FOR SUSTAINABLE DEVELOPMENT WITH MATHEMATICS TEACHING: PRESENTATION OF A TYPICAL ACTIVITY

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Education for Sustainable Development (ESD) provides valuable tools for addressing interconnected global challenges (UNESCO, n.d.). Nevertheless, there is not much “advice” in the relevant literature on “how” to integrate ESD with mathematics teaching (Hui-Chuan & Tsung-Lung, 2022). The study presented here aims to address this gap. A teaching experiment took place in a typical Greek primary school in Athens and all of its pupils (110) participated in it. During a whole school day, the children worked in groups of four on challenging but fun mathematical tasks. The course designed for the 5th and 6th Year classes of the school (10-12 years old) included 6 different activities which had “traveling” as their common theme.

A typical activity was the “How can you sit during a flight?” which aimed to introduce the mathematical concept of “combinations”. Each group was given pictures and worksheets and were asked to explore all the different ways that 3 friends can sit 3 seats during a flight. The discussions were encouraged and directed by a researcher acting as a teacher.

All the children enjoyed working on the task and used “combinations” for solving their problem. Most importantly, they exhibited important “sustainable development skills” such as “modelling with mathematics”, “evaluation of arguments” or “participation in decision making”.

ACKNOWLEDGEMENT

The financial support from the Department of Primary Education of the National and Kapodistrian University of Athens for this project is gratefully acknowledged.

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STUDYING ADVANCED MATHEMATICS THROUGH A HYBRID MOOC

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Hybrid Learning approach (HL) is an approach of blending both face-to-face and online learning settings. Recently, Massive Open Online Courses (MOOCs) have been utilized in conjunction with face-to-face traditional lessons as a novel kind of HL (de Moura et. al, 2021). Previous research mainly studied the integration of MOOCs in learning in the context of HL in higher education (Barak et al., 2016). Scant research had been done on this topic in formal high school mathematics lessons, and limited studies had examined the hybrid environment in its two settings (e.g., Hollebrands & Lee, 2020). The purpose of this study is to examine the perceptions of 55 high school teachers who teach advanced mathematics towards the integration of a hybrid MOOC in formal school lessons. It also aims to investigate this integration in practice in their classes, while exploring the combination of both the online and classroom settings of the hybrid MOOC. The study sheds light on a hybrid MOOC called "Simply Math" that aims to prepare secondary school students for advanced school mathematics. Research tools include a perception questionnaire towards MOOC integration in a hybrid format, as well as personal interviews and classroom observations for exploring the MOOC implementation in actual practice in teachers' classes. Quantitative and qualitative analyses of the results revealed common themes, indicating that from both teachers' perceptions and the observed teachers' practice in class, teaching with the hybrid MOOC has variety of benefits to teachers and students. Specifically, benefits related to class time were highly visible. For instance, about 45% of the teachers stated that it provides students with additional time for intensive practice, and roughly 57% of them appreciated that it allows individual attention for students. A theoretical contribution of the current study relates to deepening the scholarship about integrating a MOOC into formal high school mathematics learning, particularly in an HL approach, while combining the advantages of both online and classroom settings.

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A MATHEMATICS TEACHER EDUCATOR'S NOTICING

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Noticing is a skill that is an essential to effectively teach mathematics. Therefore, learning this skill is necessary for teachers. It is also expected that Mathematics Teacher Educators (MTEs) be capable to notice actively in order to teach noticing. Noticing skill of MTEs requires both being able to notice the student's thinking and noticing the teacher's thinking (Amador, 2022). Little is known about MTEs' capability to notice. The aim of this study is to reveal how the noticing process of the MTEs takes place.

The data collection tool of this research, which has a qualitative research design, is the researcher's field notes and reflective diaries. The MTE who conducted this research has over 20 years of experience teaching in both public schools and math education programs. In this study, the MTE spent 48 hours observing 12 PTs as they practiced their teaching in a real classroom setting. The MTE actively used the noticing skill while taking comprehensive field notes to engage with the students' thinking that occurred during PTs' teaching practice and what the PTs did .The MTE's field notes and reflective diaries were analyzed descriptively.

An example of the field notes from MTE: Why is 50% half? PT asked the students. Students could not say anything. The PT did not think that the student would have difficulties at this time. It appears that the student was unable to make connection between the percentage representation and the quantity of fractions. The PT asked the students to switch from percentage to fraction, which caused the student to struggle. At this point, the PT needs to establish the relationship that all these representations show half, even if they appear to be different.

The MTE had the opportunity to develop suggestions for mathematics teaching while capturing students' thinking by establishing the relationship between PTs' teaching practices and students' learning. Taking comprehensive field notes can be used by the teacher trainer to practice the noticing skill.

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EXPLORING A LESSON STUDY FRAMEWORK

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One of the fundamental purposes of education is not simply to ensure that students are taught but to ensure that they learn. If teachers take the time to investigate what their learners are learning, their professional development can be more specifically targeted to improve their lessons and assessments. Lesson study (LS) plays a promising role in deepening teachers' PCK, understanding and developing teachers' teaching skills, and teachers' ability to observe and understand student learning (Gonzalez, Villafane-Cepeda, & Hernandez-Rodriguez, 2023; Venketsamy, Hu, Helmbold, & Auckloo, 2022). This research synthesis explores a conceptual framework for lesson study implementation through analysis of research projects conducted in the United States. Socio-cultural factors affect the design, implementation, and assessment of lesson study research. This conceptual framework is better presented visually through a research poster format to illustrate the different aspects of the framework analyzed through socio-cultural paradigm.

Collaborative and school-wide lesson study is not a common practice in most countries including the United States. There may be more short-term efforts to experiment with LS implementation but the efforts are not sustainable. The disparity in terms of focus on lesson study involving preservice teachers and that of in-service teachers are dependent on their differing levels of teaching experiences and lesson design backgrounds. Moreover, K-12 school and university partnerships are also limited to districts where external funding is secured to support short-term LS collaborations. Although most short-term implementation capitalizes on existing school to university partnerships, maintaining and sustaining LS as a systemic approach to learn collaboratively continues to be a challenge. It is important to examine LS implementation through a socio-cultural lens.

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CARTOON-BASED VIGNETTES AS A STIMULUS FOR REFLECTION

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According to various models, *reflection* is a process of looking back at situations to gain insights about future actions and practices (Karsenty & Arcavi, 2017). In this study, pre-service teachers (PSTs) reflected on classroom situations presented in cartoon-based vignettes. Vignettes in cartoon format can profit from the design advantage that the complexity of classroom situations can be controlled relatively well (e.g., Herbst et al., 2011). However, the use of vignettes as a stimulus for reflection is still scarce in research. Therefore, the goal of our study is to examine *which categories of reflection can be identified when PSTs reflect on vignette-based situations?*

At the base of this study are three cartoon-based vignettes, each present a situation from an algebra or geometry classroom in junior high-school (7^{th}-9^{th} classes). 26 German PSTs analysed the vignettes according to a set of guiding questions. Their answers were analysed with interpretive methods, using categories of reflection (actions and phases which are part of a comprehensive reflective process).

Various categories of reflection were identified: The PSTs analysed the mathematical assignment, the students' thinking and the teachers' reactions, while evaluating the teachers' actions, referring to the vignette-teachers' possible goals, as well as to their own goals and views, considering different aspects of teaching and learning and relating to challenges of teaching. They also related to possible future actions (‘what would I have done’), while offering alternative practices and actions and considering their possible implications.

We conclude that the cartoon-based vignettes offer opportunities for reflection. Moreover, the results indicate that (1) a *rich mathematical assignment* in the vignettes played a key role for eliciting the PST’s mathematical and pedagogical analyses of the situation, (2) an *'open-ended' vignette dialogue* between the teacher and the students led the PSTs to evaluate the teacher's actions and to asses alternatives, and (3) the *guiding questions* 'pushed' the PSTs towards reflection, as it compels them to look not only backwards on the situation but also forward into what they would have done.

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MEASURING LONGITUDINAL PERFORMANCE AT THE MATH MINDS INITIATIVE: AN INTERVENTION AT THE ELEMENTARY LEVEL

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The Math Minds Initiative intends to improve mathematics instruction at the elementary level through design-based research (Cobb et al., 2003) in collaboration with school districts, academics, and a teaching resource developer. A model for effective mathematics teaching was developed in the first stage of the initiative (2012-2017) informing the next steps in the project. Data during this stage included: weekly mathematics lesson observations; video recording (about 300 lessons); teacher (44) and student (228) interviews; and longitudinal analysis of student performance in mathematics (n=409), as measured by the Canadian Test of Basic Skills (CTBS) (Nelson Education, 1988). This poster discusses the model used for longitudinal data analysis, its implications, and its limitations. Further information about the teaching model and other results from the initiative can be found on the Math Minds website (https://www.structuringinquiry.com).

To accommodate for missing data over time due to new enrolment and attrition, we decide to use a linear mixed model (West, 2009) to analyze changes in student CTBS scores in mathematics. Results showed a statistically significant ($p<0.05$) improvement in both the total scores and the scores per component—concepts, problem-solving, and computation—for the three schools involved in this stage of the study. These results suggest that the intervention associated with this initiative resulted in an improvement in student mathematical skills.

While the linear mixed model accommodated for missing data, there was still an effect on the means of the test scores as some students leave and new students entered the project over the years: Students leaving the project had usually shown some improvement while students new to the project often entered with a low performance. Thus, showing just the means of student scores could underestimate student growth. Another limitation was that when analyzing single classrooms, the number of students was reduced and the analysis, for most of the cases, where not statistically significant.

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SELF-REGULATED LEARNING WHILE SOLVING MATHEMATICAL PROBLEMS: EXAMINING DIFFERENCES BETWEEN MATHEMATICALLY GIFTED AND TYPICAL ACHIEVERS' STUDENTS

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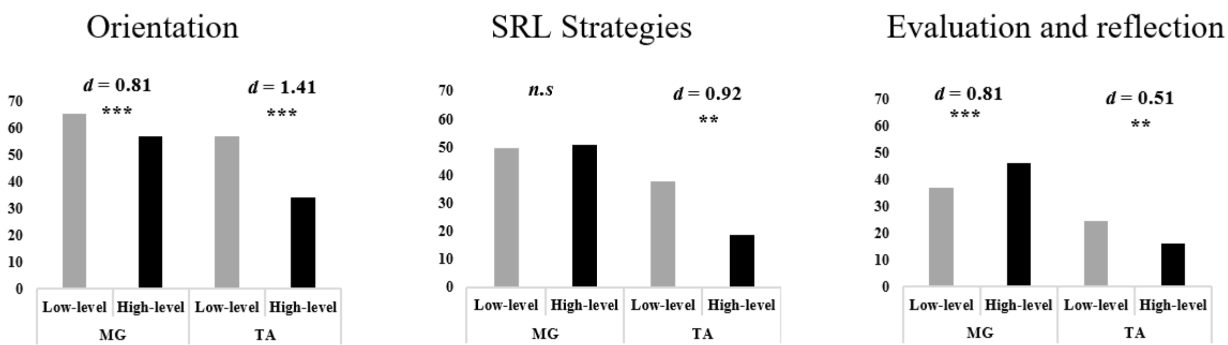
Self-regulated learning (SRL) represents a necessary fundamental 21st-century skill for children and adolescents, and is essential for the development of gifted students. This study explored mathematically gifted (MG) junior high school students' SRL capabilities while solving mathematical problems compared to typical achievers (TAs). To date, few studies have compared MG and TA students' use of SRL strategies during mathematical problem-solving. A sample of 71 ninth-grade students from three junior high schools were divided into two study groups (MG and TA). Participants completed Motivated Strategies for Learning Questionnaires and a Metacognitive Strategy Usage Questionnaire. In addition, students underwent Think-Aloud Interviews while solving mathematical problems (quadratic function) at two levels of difficulty. The results only revealed significant differences between MG and TA students on the organization subscale of the MSLQ questionnaire. No significant differences were found regarding the other three subscales (i.e., critical thinking, metacognitive self-regulation, and time and study environment management). Additionally, no differences were found between the study groups in the metacognitive strategies used questionnaire. MG students outperformed TA students on the mathematical problem-solving test and generally used more SRL strategies during the qualitative think-aloud interviews. Moreover, the between-group differences in effect sizes were higher on the high-level problem-solving task than the low-level difficulty task. The findings constitute a significant milestone in the study of SRL capabilities of MG in junior high school.

Figure 1: Mean (and S.E) of Percentage of SRL Strategies Used by Group and Problem Difficulty Level



GEOMETRIC REASONING USED IN CHILEAN MATHEMATICS TEXTBOOKS: THE CASE OF SEVENTH AND EIGHTH GRADE

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The geometry is one of the aim topics along the mathematical scholar curricula where the geometrical reasoning is the main process that promote the learning of mathematics. This reasoning is developed by students when such abilities are considered for the solution of geometric problems with distinct cognitive demand (Hershkowitz et al., 1998). In the scholar context, the textbooks are the vehicle to effectively promote knowledge, but they must not be considered as the unique source of geometric knowledge, especially when they consider geometric problems with low cognitive demand (Shield and Dole, 2013). Given the relevance of the development of geometric reasoning and the use of scholar textbooks as fundamental tools in the classroom, the purpose of this short oral is the analysis of the aim characteristics of the geometric reasoning that promote the textbooks. We conducted a qualitative content analysis of two mathematical scholar textbooks (7^{th} and 8^{th} grade), which are provided by the Ministry of Education, currently in force, for middle-high school students (between 12-13 years of age) in Chile.

The analysis show that from 373 geometrical problems, the 90% of them promote only the attribute recognition (shape and measure) and the measure associated to magnitudes of geometrical figures. The problems of the scholar textbooks bring numerical data that allow the student to develop procedures that involve the use of geometrical formulas and geometrical figures as illustration, which exhibits a detriment of the geometrical reasoning. Here, the knowledge leaf behind intrinsically processes of the geometry which are fundamental to enrich the geometrical reasoning by the use of arithmetic-algebraic processes. As there is a quietly relation between algebraical and geometrical reasoning, it is suggested to extend, as much as needed, the geometrical reasoning during the resolution of geometric problems using properly process of the geometry (visualization, deduction and construction), indeed to use algebraic processes.

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DEVELOPMENT AND EVALUATION OF OPEN EDUCATIONAL RESOURCES TO IMPROVE TEACHERS' KNOWLEDGE ON SPATIAL ABILITIES

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Open Education Resources (OER) as digitized learning environments are arranged for non-commercial purposes such as reuse and adoption. OER initiatives in teacher education emphasize positive impacts through innovation resulting from free exchange of expertise and successful course concepts (Ramoutar, 2021). A course concept that increases motivation and growth is the Inverted Classroom Model (ICM). ICM disrupts conventional course structures by engaging students to delve into the subject in individual learning phases prior to the classroom. In shared learning phases students then apply and discuss what they have learned (Altemueller et al. 2017). Spatial abilities are important in mathematics learning. Research projects focus on the modelling and improvement of spatial abilities, children's strategies to solve spatial tasks, or questions of diversity in classroom (Heil, 2021). This research project aims to implement these foci in OER learning environments, to integrate them in an ICM course, and to investigate students' subjective experiences in this course.

The samples comprised two Bachelor courses that tested the ICM format in 2022/2023. The students completed a questionnaire concerning their overall motivation to participate, their subjective experiences in the individual learning phase, and their subjective experiences in the shared learning phase. They also replied to open questions concerning possible implications of the ICM format on their learning process. First results show that the students experienced the course as a challenging, yet self-directed, motivational, and collaborative format of learning about spatial abilities. Further research, however, is required to investigate individual learning trajectories and interactive processes in greater detail. The poster presents the overall ICM with the thematic foci that where identified, illustrates learning environments and their OER implementation, and addresses open questions.

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WHATSAPP GROUP + BAGRUT = BAGROUP: TEACHERS' PERSPECTIVES

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The most commonly used social networking sites is WhatsApp. Recently, the use of social media has also begun to spread into educational systems. However, only about 40% of teachers use social networking sites often or always for educational purposed (Calderón-Garrido & Gil-Fernández, 2022).

Our Poster Presentation will address study investigated teachers' perspectives about what opportunities for learning and teaching could be created using WhatsApp as a social network to help students prepare for the final secondary-school Bagrut (matriculation) exam in mathematics. Launched by the Ministry of Education and the Center for Educational Technology three months before the Bagrut examination, the "Bagroup" project was initiated to serve as an additional environment for learning mathematics. The formation of 40 WhatsApp groups was meant to provide an online review project during which high school students around Israel who are not familiar with each other, as well as with their teachers, presented problems with which they were having difficulties. The study used a mixed method, sequential explanatory procedure. Data were collected using three tools: a questionnaire with Likert-type (1-5) statements and open questions, informal semi-structured interviews, and observations of four Bagroup study groups conducted during the three-month period. Factor analysis revealed three categories regarding the Bagroup environment: five factors that contribute to learner's emotional needs, four factors that promote learning, and two factors that inhibit learning. The teachers perceived that learning in the Bagroup framework meets the policy recommendations of the OECD (2016) for making mathematical knowledge (both at individual and group levels) available to all, and equal opportunities for all to ask and express ideas.

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MATHEMATICS ANXIETY AND NUMBER SENSE IN CHILDREN: AN EVENT-RELATED POTENTIAL STUDY

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Background: Math anxiety is defined as "a feeling of tension, apprehension, or fear that interferes with math performance" (Ashcraft, 2002). As anxiety is regulated, students often see a marked increase in their math performance.

Methods: We recruited forty 5th grade students. We investigated and compared the degree basic numerical skills affected by math anxiety between the high-math-anxiety (HMA) and low-math-anxiety (LMA) groups. The neural underpinnings of arithmetic cognition in children with math anxiety was explored by the event-related potential technique when they processing basic numerical processing tasks.

Results and Discussion: Results indicated that the HMA group had lower accuracy than the LMA group in a subitizing task. ERPs measured indicated that the HMA group had a smaller P2p and P3 amplitudes in a subitizing task and more delayed P3 latency in an approximating task compared with the LMA group. In the symbolic numerical comparison tasks, the HMA group responded more slowly and had a delayed P3 latency when processing paired Arabic digits. Our findings revealed that the HMA children might have less precise representations of numerical magnitude than their LMA peers.

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LONGITUDINAL IMPACT OF MATHEMATICS NEWS SNAPSHOTS ON STUDENTS' VIEWS OF MATHEMATICS

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The study focused on characterizing the longitudinal impact of the Mathematics News Snapshots (MNS) project on high-school students' perceptions of mathematics as a discipline and as a profession.

METHODOLOGY: Data collection was carried out between 2016-2019. Participants were high school students (grades 10-12) who took part in the MNS project, and students from parallel classes in the same schools as a comparison group. Data sources included student questionnaires, student interviews, and teacher feedback. Data analysis involved both qualitative and quantitative tools.

FINDINGS: Findings indicate that students who participated in the MNS project tended more each year (and more than the comparison group) to describe mathematics as a dynamic, creative, and valuable discipline, regarded the existence of recent results as well as of open problems, and recognized its applicability to our daily lives. Further, our results indicate that throughout the duration of the MNS project, the experimental group showed an increasing understanding of the collaborative nature of modern mathematicians' work – compared with the comparison group, who tended to view mathematicians as solitary and pedantic.

Our results indicate that the Snapshots project has contributed to shaping student conceptions about mathematics as a discipline and as a profession. While the literature documents misconceptions of mathematics as deterministic body of facts and procedures that are at best loosely connected to each other or to the real world, and as a solitary activity (Op't Eynde et al., 2002), the experimental group showed an improved viewpoint of the true nature of mathematics and mathematicians' work compared with the comparison group.

In conclusion, conceptions about mathematics do not develop in a vacuum and depend greatly on the student's school learning experiences. By incorporating Mathematics News Snapshots into the day-to-day school experience, over the course of three years of high school, we showed that some of these conceptions can change.

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ON THE REASONING OF PRE-SERVICE TEACHERS WHEN SOLVING OPEN-ENDED GEOMETRIC PUZZLES

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Open-type problems provide an effective way to explore geometrical objects and their properties (e.g., Munroe, 2015). Because prospective teachers often have difficulty finding the very initial solutions, tasks that enable some immediate answers could be a tool to help them. About 60 pre-service teachers dealt with two open-ended geometric puzzles, each of which having a large (potentially infinite) set of solutions. These were i) sectioning a regular polygon into equal-area pieces using lines originating in the centre, and ii) constructing polygons with integer areas using 12 one-unit segments.

For the first task, all the students came up with various divisions of the polygon into *congruent pieces* but found it difficult to progress to the general partitioning into *pieces of equal area*. However, with the forced use of GeoGebra, 40% of the students detected an invariant: if the parts of a perimeter line are equal in length, the respective pieces are equal in area. This leads to an *infinite set* of partitions for each number of pieces.

In the second task, every student constructed at *least three types* of the following solutions: rectangles, 3-4-5 triangle, rectilinear polygons as rectangles with “pushed in” one-unit squares, polygons as rectangles with two sides replaced with congruent regular triangles, and parallelograms as “inclined rectangles” with an integer value of the height. Only two students (they made use of GeoGebra) discovered additional types of polygons and realized how to construct an infinite *continuous set of solutions*.

The discussed multi-directional tasks led students to *various scenarios*. They were able to construct relevant *multiple* solutions based on “problem solving/problem posing” cycles, using (intentionally or intuitively) *hidden invariants* (Sinitsky & Ilany, 2016). However, students had difficulty transitioning to *another type* of solution, and typically *varied* the open-ended task rather than exploring *all the solutions* within the given task. They rarely used a dynamic geometry tool of their own volition. Yet, the incorporation of GeoGebra critically influenced the space of solutions they reached; some students stated that the *dynamic manipulation* brought them a “breakthrough discovery.”

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AN ACTIVITY THEORY PERSPECTIVE ON STUDENT-REPORTED CONTRADICTIONS TO UNDERSTAND COLLABORATION DURING PROBLEM SOLVING IN PRIMARY SCHOOL MATHEMATICS.

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Collaborative problem solving (CPS) is a powerful pedagogy within mathematics education as it can promote other competencies of importance such as metacognition (Goos et al 2006). Those with better collaboration and metacognitive skills make more contributions to the problem solving processes. However, those who do not have sufficient communication skills may be disadvantaged and not able to learn to the same extent as their more competent peers. Whilst much research in collaborative learning has focussed on optimum group composition for successful solving of problems, less attention has been paid to the specific social processes that learners encounter. There is limited evidence understanding these processes from the perspective of learners. The aim of this case study was to use Activity Theory (Engestrom 2009) as a framework to understand student perceptions of communication in CPS during primary school mathematics lessons. Fourteen students aged 9 & 10 were video recorded over a number of sessions while engaging in CPS in the naturalistic setting of their classroom. A multi-method approach of video of interactions, critical event recall and activity systems analysis resulted in an in-depth understanding of the reasons for students choosing to, or choosing not to, interact with their peers.

Critical events were identified in the problem solving sessions, where it was deemed that students made decisions which altered the outcome. They were then asked to discuss what they were thinking at the time and what might have shaped their decisions. Results highlighted three areas in the Activity system where contradictions occurred and subsequently had an impact on the interaction patterns. These were rules, mediating artefacts, and division of labour. Furthermore, whether groups were successful or not in their problem solving, a perceived holder of knowledge emerged. Importantly, once the role was assigned, students found it difficult to deviate from this which impacted the direction of the problem-solving outcome.

These data add to the evidence that quantifying levels of collaborative interaction is not sufficient for understanding its relationship to successful outcomes. Rather, the social processes that lead to these interactions should also be considered.

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ELEMENTARY STUDENTS' (IN)FLEXIBLE STRATEGY USE IN WORD PROBLEMS AS REVEALED BY EYE-TRACKING

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Students' mathematical thinking and performance are of critical importance in their school success. Elementary students are strongly inclined to follow prescribed patterns or algorithms (Csíkos & Szitányi, 2020) during word problem solving. Different solution strategies applied by students can be detected by various methods including eye-tracking (Strohmeier et al., 2020).

Our research questions were (1) What are students' choices when they are offered with different types of answers to a realistic word problem? (2) How (and whether at all) do students modify their choice while solving a series of similar tasks?

21 students took part in our eye-tracking data collection in their 4th grade of schooling. A realistic word problem was administered with seven consecutive solutions in an on-line platform. On each page, the text of the realistic task appeared first together with a helpful pie chart, and the students had to decide whether the offered solution was right or wrong. Students could jump back and forth amongst the different (textual versus algebraic and computationally correct versus incorrect) solutions. A mobile eye-tracker was used to gather information about eye-fixation times and areas.

Although the majority of the students found the realistic answers appealing, and judged them as correct, only four of them went back to any of the previous pages to reevaluate any solutions, and only one student did actually modify one of their original answers after reading and judging the realistic answer. Based on the eye-tracking analysis, two thirds of the students had fixations on the helpful pie chart when reading the task for the first time. In the six subsequent solutions, this ratio dropped to 0-20%. The results bring further evidence of students' inflexible strategy use on a realistic word problem. The lack of fixations on the helpful pie chart is alerting, considering that the original Dutch term "zich realiseren" (from which the adjective 'realistic' was derived) is related to imagination.

This work was supported by the Research Programme for Public Education Development of the Hungarian Academy of Sciences

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PRESERVICE TEACHERS' PEDAGOGICAL REASONING FOR INTEGRATING DIGITAL GRAPHING TOOLS IN EXPONENTIAL AND LOGARITHMIC FUNCTIONS

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Mathematics teachers' pedagogical reasoning has received growing research attention. Additionally, technology integration in mathematics classes has become increasingly prevalent. This study aims to explore preservice teachers' pedagogical reasoning for integrating digital graphing tools in teaching exponential and logarithmic functions.

This study used a questionnaire with three teaching vignettes to collect the data from 91 preservice teachers (PTs) in two normal universities in Taiwan. The vignettes depicted teaching situations in which the digital graphing tools, such as GeoGebra, Desmos, and Excel, were used to achieve various teaching objectives — exploring features of exponential/logarithmic function graphs (Vignette 2), validating the change of base formula (Vignette 4), and modelling real situations using exponential functions (Vignette 3). For each vignette, PTs were asked seven questions, such as, please evaluate this teaching situation from the perspectives of integrating technology and provide reasons. Toulmin's (1978) reasoning framework, consisting of claims, grounds, and warrants, was adopted for a content analysis on the PTs' responses.

Most PTs (54%~86%) claimed that integrating digital graphing tools can help develop mathematical knowledge and competence in students, while few (3%~10%) mentioned potential impact on students' affective aspects. Most PTs' grounds to support their claims are pertinent to developing student competence (78%~85%). The following most frequently mentioned grounds are about student learning activities (62%~78%) and the functional capability of technology (59%~75%). In Vignette 4, more than 1/3 of the PTs' grounds are related to the nature of mathematics, whereas the percentages are less than 10% in the other two vignettes. The PTs argued that the equivalence of $\log_a x$ and $\frac{\log x}{\log a}$ cannot be validated by having the same graphs but instead requiring an algebraic reasoning process. Most PTs' warrants were about re-balancing emphasis on skills, concepts, and applications (69%~81%) and changing classroom and didactic contracts (71%~84%; Pierce & Stacey, 2010).

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EXPLORING AN ALTERNATIVE ANALYTIC FRAMEWORK UNDER THE HOLISTIC PSYCHOLOGY OF EMOTION AND COGNITION

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National Taiwan Normal University

The holistic psychology of emotion and cognition is a trend in mathematics education (Hannula, 2001; Roth & Walshaw, 2019). Roth and Walshaw point out that very few analytic framework can be used to analyse students' learning under the trend (Roth & Walshaw, 2019), so they propose multi-perceptual drama metaphor as their framework. Besides that, Hannula (2001) posed out four meta-level constructs, meta-cognition, meta-emotion, emotional cognition and cognitive emotion, to analyse students' learning. When analysing the holistic teaching video, such as rectangular number game (Chu & Lin, 2019), both frameworks cannot analyse the driving force of the task evolving. This leads to the question that how to explore an alternative analytic framework under the trend?

We use quantum-inspiration as our method. In our framework, we use Duval's (2017) semiotic theory to analyse students' cognition. In the theory, each representation (qubytes) contains meaningful units (qubits). Similarly, we pose out the metaphor of emotions. Entanglement is a between-qubit action, if one qubit is entangled with another one, then its behavior will influence the entangled qubits, such as emotional cognition and cognitive emotion. Superposition between emotion and cognition are that the non-entangle emotion and cognition will both influence students' behaviors. Finally, when dealing a problem with qubits, circuits served as functions are needed. During students' learning, students' thinking and process, and the activity serve as the function. Based on the metaphor, we propose an alternative analytic framework.

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PEREZHIVANIE AS A BRIDGING CONCEPT FOR VYGOTSKYAN AND PHENOMENOLOGICAL LENSES

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This study uses a learning episode and its radical repercussions for a prospective teacher of secondary mathematics education, who radically altered her learning and teaching attitudes after this session. The empirical episode offers an exemplary case of the application of the so far untranslated concept of perezhivanie to mathematics education research. The author advocates for the use of perezhivanie as a suitable concept for the **integration** between the phenomenological analytical and methodological lens with the Vygotskyan approach to Activity Theory.

The starting point of the study's analysis is the intuitive moment that marked Diana's stressful learning experience. This experience was the first in these sessions to prove her own convictions right, even though they were different from anyone else's in her class. This transformation resulted in a radical change in her learning and teaching attitudes. Diana achieved this transformation in a session where she went from desperation to the *noematic* survival of sense due to her *algorithmic* intuition of *essence. "Perezhivanie"* is a concept developed by Vygotsky as a psychological and *developmental unit* of personality, in a “superior approach to that subsequently adopted by Activity Theory” (Blunden, 2016). It involves significant situations in one’s lived experience that become units of one's personality, involving a whole new perspective. Vygotsky (1934) explores a typical perezhivanie case where three children, under the supervision of an unstable alcoholic mother develop in a radically different manner, as they face their mother’s threat; the eldest child consolidated perezhivanie, as he managed to deal with the problem and helped protect the other children. In perezhivanie “personal and situational characteristics” are represented (p. 342). The study shows how the phenomenological lens may shed new light on the appearance of, and interaction between, personal and situational characteristics. The empirical example serves as a solid exemplary case for bridging the late Vygotskyan lens, which is responsible for identifying and delineating the prismatic, unit-like perezhivanie experience, and the phenomenological lens, which reveals the constitutive potential and contribution of the personal perspective.

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WHY DO REAL ANALYSIS? LECTURE 'WHY' STORIES AND TENSIONS

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Real Analysis (RA) is a difficult undergraduate course. A key challenge is the absence of compelling explanations for why RA is worth studying (Dawkins & Weber, 2017). In particular, little is known about how instructors approach the task of motivating RA in their lectures. Here, we report on an ongoing analysis of the 'why' stories for RA offered by two mathematicians, Alex and Cai, in their introductory RA lectures.

Data consist of Zoom recordings of RA lectures taught in a large R1 university in the United States in Fall 2020. The study is broadly informed by sociocultural theories. Our research questions are: how do instructors describe and justify RA, and, what tensions do these narratives give rise to? To answer them, we watched the lectures (n=2x3), created episode-segmented transcripts, and coded the transcript for explicit discussions of what RA is and why it is worth doing. We also flagged moments exhibiting tension, and related them to the issues identified in the coded 'why' stories.

We found four 'what and why RA' stories: (1) RA is Calculus with proof. Since Calculus is known, RA is a good course for learning to prove. (2) Calculus is a tool that sometimes breaks. RA is a theory of how the tool works. This theory is good for future use of the tool. (3) Mathematicians made mistakes using Calculus. They found a solution in rigor. We follow their efforts. (4) RA is a theory that builds Calculus from foundations. The purpose is to make Calculus simple and elegant through connections.

Story (1) is oriented to future learning. It offers no external why, and proof remains an arbitrary skill to master. Story (2) is pragmatically oriented to applications, whereas (3) frames RA as an opportunity to connect with a community and its history. In both stories students are tasked with suspending disbelief as the motivating problems are not apparent until the very end of most RA curricula. Story (4) frames RA as an achievement of epistemic coherence. While potentially compelling, just *telling* that something is elegant may not be sufficiently convincing for students. Episodes flagged for tension illustrate how these issues give rise to problematic demands such as asking students to *pretend* that they do not know, accept seemingly *arbitrary* norms like "sometimes it's better to give a rigorous definition," and adopt *dehumanizing* language practice such as "imagine you are talking to some alien" or a "logic machine."

Findings from this study highlight tensions to look out for when explaining RA, and can inform intentional crafting of more compelling and realistic 'why's for the subject.

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NUMBER INTERVIEWS AS ASSESSMENT TOOLS FOR REVEALING MISCONCEPTIONS IN NUMBER

Natasha Ziebell, Cath Pearn

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The steady decline of Australian results in mathematics at both international (PISA and TIMSS) and national levels (NAPLAN) have been well-documented and publicised. Teachers work under increasing expectations that they will generate and use good quality data to track individual student learning and design their teaching to meet each student's identified needs. Interview assessments can be used to determine students' current and potential mathematical understandings, revealing why a student is performing as observed or assessed by illuminating cognitive processes (Ginsburg, 1997). This research project investigated how Number Interviews assessments can be used to identify misconceptions and inform targeted planning to meet students' learning needs. These interviews were developed to specifically identify inefficient methods or strategies that students use for mental and written computations and any misconceptions or difficulties they have developed.

The research project was conducted as part of the South Australian Department for Education Learning+ online mathematics tutoring initiative. Over two years, the Number Interviews were conducted by 277 qualified teachers with 2890 students in Years 6, 7, 8, and 9. The results focus on a detailed item analysis of 13 Year 7 pre and post Number Interview results that were video-recorded and conducted during a program of tutoring that occurred twice a week over a ten week period. The results showed that a key feature of interview assessments is prompting students to explain their thinking processes, which provides unique insights into their understanding of mathematics. This allows teachers to identify what the student knows and what they are ready to learn next. Using an interview approach to ask how or why a student reached a correct answer revealed that despite being computationally correct, in some cases student reasoning is inaccurate. On the other hand, an error can be made by a student who has sound conceptual knowledge and reasoning. These understandings are difficult to be derive from traditional pen and paper, multiple choice tests. The interviews also exposed a range of deeply ingrained common misconceptions that confirms and extends previous research on topics in Number. The teachers demonstrated how they used the Number Interview results to inform targeted planning for learning that addressed these misconceptions.

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NATIONAL PRESENTATION

PME-46

MATHEMATICS EDUCATION

FOR GLOBAL SUSTAINABILITY

NATIONAL PRESENTATION:
RESEARCH IN PURSUIT OF IMPACT ON PRACTICE IN ISRAEL

INTRODUCTION

Michal Ayalon and Roza Leikin
University of Haifa

In line with the "Mathematics Education for Global Sustainability" topic of the PME-46 Conference, this national presentation is devoted to the impact of mathematics education research on practice in Israel. Connections between theory and practice in Mathematics Education in Israel have a longstanding history (Movshovitz-Hadar, 2018). The PME community in Israel has been consistently involved in changes to educational practices, framing of national curricula, and the development of innovative instructional materials and technologies (Friedlander, Even, & Robinson, 2018; Hofstein et al., 2021; Kohen & Nitzan, 2022; Leikin, 2019; Levenson et al., 2022; Naftaliev & Yerushalmy, 2017; Segal et al., 2017). Researchers from Israeli institutions of higher education are actively involved in the preparation of future generations of mathematics teachers and teacher educators as well as life-long learning of in-service mathematics teachers (Ayalon & Wilkie, 2020; Even, Artstein, & Elbaum-Cohen, 2018; Cooper & Koichu, 2021; Kohen & Borko, 2022; Leikin, 2020; Levenson, 2022; Naftaliev, 2018; Tsamir & Tirosh, 2009).

Some activities are supported by the Ministry of Education, such as the National Centers for Mathematics Teachers; others are part of design experiments supported by research foundations; and others are sponsored by private foundations. During the past decade, two centers – the National Center for Primary School Mathematics Teachers and the National Center for Secondary School Mathematics Teachers ─ operate in parallel and advance corresponding areas of mathematics teaching and learning in the University of Haifa under the academic supervision of Michal Yerushalmy and later Roza Leikin (Talmon et al, 2018). The Primary School Center has operated in the University of Haifa for more than 20 years, and was initially established by Pearla Nesher as an academic unit at the University of Haifa Faculty of Education. The National Center for Secondary School Mathematics Teachers has been managed by the University of Haifa team for about 10 years and was preceded by centers for mathematics teachers at the Technion (Kesher-Cham led by Nitsa Movshovitz-Hadar) and at the Weizmann Institute (MANOR program led by Ruhama Even (Even, 2008)).

The Trump Foundation (not connected to the former USA President) is a philanthropic foundation that has strongly influenced the impact of research on practice in mathematics education in Israel. This involvement was motivated by "A start-up nation at risk" (Hurvitz, 2018). The Trump Foundation supports programs for professional development of mathematics teachers and teacher educators, mainly through the engagement with advanced mathematical thinking in the communities of practice

(Handelman & Kohen, 2022; Koichu, Cooper & Widder, 2021; Leikin & Aizik, 2020; Segal et al., 2020), promotion of teaching high-school mathematics at high level through innovative instructional design (Cooper et al., accepted; Leikin, 2019; MesiMatica; Movshovitz-Hadar & Edri, 2013; Rotem & Ayalon, 2023; Tirosh et al., 2019), and enhancement of mathematical literacy in middle school mathematics (Cohen-Nissan & Kohen, 2023; Think Far with Mathematics; Leikin et al, in press).

Overall, there are dozens of projects and programs managed by the research community of mathematics education in Israel. During this session, examples of different ways in which research pursuits shape practice in Israel will be presented in form of TED-Mathematics education talks and are described in the following abstracts.

A PROGRAM FOR MATHEMATICS AND HISTORY TEACHERS FOR INCORPORATING HISTORY OF MATHEMATICS IN SCHOOL TEACHING: COLLABORATION ACROSS DISCIPLINES

Abdelrahman Affan and Michael N. Fried

Ben-Gurion University of the Negev

Although using history of mathematics (HoM) in mathematics teaching has been a theme since the days of D. E. Smith and Florian Cajori at the start of the 20th century, criticism of it on theoretical grounds has been much more recent (Fried, 2014). Such criticism has arisen, *inter alia*, from the failure to recognize that HoM is *both* history and mathematics, that it is a venture involving *two* disciplines, each with its own set of presuppositions and practices. It thus needs to be viewed educationally under the set of ideas connected to disciplinarity, multidisciplinarity, interdisciplinarity, and, ultimately, transdisciplinarity (see Darius-Smith & Macarty, 2016, and Jao and Radakovic, 2019, for these terms). With that in mind, a program was set up involving mathematics and history teachers in the Arab sector of Israel. They studied a text by the 10th century Islamic mathematician Abu al-Wafa' al-Buzjani and worked together to design a teaching unit for students incorporating HoM. Their collaborative work was viewed in terms of the spectrum of disciplinarity to transdisciplinarity, using these as descriptions not only of activities, but also, more significantly, of *kinds of collaboration*.

WHAT CAN COUNT AS STRONG IMPACT?

REFLECTIONS ON A TEACHER-RESEARCHER ALLIANCE FOCUSING ON PROBLEM SOLVING

Jason Cooper and Boris Koichu

Weizmann Institute of Science

Arguably, any teacher professional development program aims to have an indirect impact on students' learning. Yet there is likely to be significant variation in the nature of the impact that different programs aim to achieve. We consider the impact *strong* if it retains some measure of fidelity as it is sustained over time and scaled up. In this talk we elaborate our conceptions of fidelity, sustainability, and scale, and demonstrate how we work with teachers to achieve them in one PD program – *Raising the Bar in Mathematics Classrooms*. The aim of the program is to support regular incorporation of problem solving in middle-school mathematics lessons for all students and in all segments of Israeli society. This program is organized as a network of professional learning communities, in which more than 300 teachers have taken part since 2020. Our conception of impact is situated in a theoretical-organizational model that we call TRAIL – Teacher-Researcher Alliance for Investigating Learning. TRAIL features involvement of mathematics teachers in practices and processes of disciplined educational inquiry in co-learning partnerships with mathematics education researchers. As the notions of *alliance* and *co-learning* suggest, we aim for a mutual kind of impact that is shared by researchers and teachers. Thus, teachers' experiences incorporating problem-solving activities in their lessons, and the challenges of sustainable up-scaling, contribute to our learning, which in turn helps us explicate our implicit theories of impact, and how it can be attained.

USING ASSESSMENT TO INFORM INSTRUCTIONAL DECISIONS: A THREE-DECADE RESEARCH AND DEVELOPMENT JOURNEY

Ruhama Even

Weizmann Institute of Science

This talk (based on Even (2021)) will present a three-decade research and development journey addressing the problem of using assessment to inform instructional decisions (i.e., formative assessment), highlighting:

- Three milestones in our study of the challenges associated with eliciting, interpreting and acting on evidence of student learning of mathematics. They are related to teachers' knowledge about students' learning of mathematics, the complexity inherent in interpreting students' talk and actions, and ways of addressing students' errors in mathematics.
- Three approaches we designed for addressing these challenges: the *Manor* approach, the *MesiMatica* approach, and the *Thinking Far with Mathematics* approach.

The talk will demonstrate the way the different facets of this work were (and still are) often interwoven, contributing to, and building on, each other.

MATHEMATICAL MODELLING IN THE ISRAELI EDUCATION SYSTEM

Zehavit Kohen

Technion – Israel Institute of Technology

Contemporary 21st-century skills require new ways of teaching mathematics to prepare students for the changing world, in which there is a high demand for excellent STEM (Science, Technology, Engineering and Mathematics) professionals. The skills required of excellent mathematical problem-solvers are not taught by teachers in the context of professional tasks, so students do not gain the experience and understanding of applied mathematics as it is used in STEM workplaces. Furthermore, high-school students often lack motivation to study mathematics, as they struggle to understand its relevance to the world around them. In this talk, I will describe my current research on the integration of authentic STEM-related mathematical modelling problems in secondary school math teaching. The research focuses on the design and integration of i-MAT (Integrated Math & Technology) materials in professional communities of leading teachers and of teachers who integrate these materials in their classes, as well as the effect on students' learning.

PROVIDING RICH MATHEMATICAL EXPERIENCES FOR STUDENTS WITH SPECIAL EDUCATIONAL NEEDS

Esther S. Levenson

Tel Aviv University

Inclusive education is not merely the placement of students with special education needs (SEN) in mainstream classes. At Tel Aviv University we are investigating ways of engaging students with SEN in rich and challenging mathematical activities, whether they are learning in mainstream classes or in special education classes. For example, can a third-grade student with autism spectrum disorder not only recognize mistakes in a two-digit addition computation, but can that student reason out why such a mistake might have been made? (Spoiler, he can!) In a third-grade special education class, with the help of mediation from the teacher-researcher, students engage with open-ended tasks (and find several different solutions). In the sixth grade, approximately 120 students, working either individually or in small mixed-ability groups, solve multiple-solution problems. Working with students of various ages has offered us insight into ways of including students with SENs in rich mathematics. The impact of those studies on the wider elementary school population, will hopefully be felt in the upcoming year, as a professional development program for elementary school teachers will be conducted, incorporating what has been learnt from those previous studies, with the aim of promoting teachers' pedagogical content knowledge, as well as their self-efficacy, for teaching rich mathematics in classrooms that include students with SEN.

MATHEMATICS NEWS SNAPSHOTS

Nitsa Movshovitz-Hadar and Abraham (Avi) Berman
Technion – Israel Institute of Technology

Mathematics is an active research area where many beautiful and very useful results are obtained all the time, but most students and many teachers do not know it. The Mathematics News Snapshots Project (https://mns.org.il) was developed in order to bridge over the gaps between contemporary mathematics and school mathematics, between the time it takes to follow the literature and the time teachers have, and between the image of mathematics and mathematicians that students possess and the true nature of mathematics as well as of doing mathematics.

This is done by interweaving mathematics news snapshots, each describing a cutting-edge result, its brief history, and notable relevant mathematicians. In the talk we will describe the pedagogy used in preparing the snapshots (Socratic dialogue, surprises, and more) and what was accomplished in a decade of intensive combination of development, implementation, and research. Spoiler – quite a lot.

TEAMS PLCS PROJECT

Talli Nachlieli[1] and Einat Heyd-Metzuyanim[2]

[1]Levinsky-Wingate Academic College, [2]Technion - Israel Institute of Technology

TEAMS PLCs project stands for professional learning communities that aim at Teaching Exploratively for All Mathematics Students. In our PLCs, we engage over 300 secondary school mathematics teachers from all parts of Israeli society, in learning and developing teaching practices. Those practices aim at providing students with opportunities for explorative participation, that is, with opportunities to think, collaborate, and communicate their ideas. Our teachers work on exploration-requiring tasks for which the students do not have ready-made procedures to apply, and which are appropriate for the Israeli secondary school curriculum. We work with teachers to conceptualize teaching practices that support productive discussions by developing artifacts such as monitoring sheets and video excerpts.

EXPERIMENTAL MATHEMATICS AT SCHOOL: LEADING TEACHERS TOWARDS A CHANGE

Elena Naftaliev and Marita Barabash

Achva Academic College

Experimenting within mathematics at school – why and how? Is it compatible with the curriculum? Are the teachers apt to it, and if they are not - how can we help them adjust to the idea of intertwining experimental mathematics with formal? Information technology enhances mathematics as an empirical discipline, adding to the tension of the empirical-inductive vs. formal deductive approaches to the subject. This tension manifests itself in mathematics teaching and learning. We study ways of inducing pre-service and in-service teachers to enrich their orientations for educated integration of these approaches. We will present what we have learned from our long-term design and research project and what is there more to learn.

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